Möbius geometry of three-dimensional Wintgen ideal submanifolds in $S^5$

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Abstract Wintgen ideal submanifolds in space forms are those ones attaining equality at every point in the so-called DDVV inequality which relates the scalar curvature, the mean curvature and the normal scalar curvature. This property is conformal invariant; hence we study them in the framework of Möbius geometry, and restrict to three-dimensional Wintgen ideal submanifolds in $S^5$. In particular, we give Möbius characterizations for minimal ones among them, which are also known as (3-dimensional) austere submanifolds (in 5-dimensional space forms).

Keywords Wintgen ideal submanifolds, DDVV inequality, Möbius geometry, austere submanifolds, complex curves

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1 Introduction

The so-called DDVV inequality says that, given an $m$-dimensional submanifold $x : M^m \rightarrow \mathbb{Q}^{m+p}(c)$ immersed in a real space form of dimension $m + p$ with constant sectional curvature $c$, at any point of $M$ we have

$$s \leq c + \|H\|^2 - s_N.$$  \hspace{1cm} (1.1)

Here $s = \frac{2}{m(m-1)} \sum_{1 \leq i < j \leq n} \langle R(e_i, e_j) e_j, e_i \rangle$ is the normalized scalar curvature with respect to the induced metric on $M$, $H$ is the mean curvature, and $s_N = \frac{2}{m(m-1)} \| \mathbb{R}^2 \|$ is the normal scalar curvature. This remarkable inequality was first a conjecture due to De Smet et al. [8] in 1999, and proved by Ge and Tang [9] and Lu [16] in 2008 independently.

As pointed out in [6,8,16,17], it is a natural and important problem to characterize the extremal case, i.e., those submanifolds attaining the equality (1.1) at every point, called Wintgen ideal submanifolds. In [9] it was shown that the equality holds at $x \in M^m$ if and only if there exist an orthonormal basis

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\{e_1, \ldots, e_m\} of \textit{T}_x M^m \text{ and an orthonormal basis } \{n_1, \ldots, n_p\} \text{ of } T_x^\perp M^m \text{ such that the shape operators } A_{n_r}, r = 1, \ldots, m \text{ have the form }

\[
A_{n_1} = 
\begin{pmatrix}
\lambda_1 & \mu_0 & 0 & \cdots & 0 \\
\mu_0 & \lambda_1 & 0 & \cdots & 0 \\
0 & 0 & \lambda_1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \lambda_1
\end{pmatrix}, \quad A_{n_2} = 
\begin{pmatrix}
\lambda_2 + \mu_0 & 0 & 0 & \cdots & 0 \\
0 & \lambda_2 - \mu_0 & 0 & \cdots & 0 \\
0 & 0 & \lambda_2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \lambda_2
\end{pmatrix},
\]

(1.2)

and

\[A_{n_3} = \lambda_3 I_m, \quad A_{n_r} = 0, \quad r \geq 4.\]

(Here, \(I_m\) is the identity matrix of order \(m\).) This is the first step towards a complete classification.

Wintgen [21] first proved the inequality (1.1) for surfaces \(M^2\) in \(\mathbb{R}^4\), and that the equality holds if and only if the curvature ellipse of \(M^2\) in \(\mathbb{R}^4\) is a circle. Such surfaces are called \textit{super-conformal} surfaces. They come from projection of complex curves in the twistor space \(CP^3\) of \(S^4\) (see [3]). Together with totally umbilic submanifolds (spheres and planes), they provide the first examples of Wintgen ideal submanifolds. Note that they are not necessarily minimal surfaces in space forms. In particular, being super-conformal is a conformal invariant property, whereas being minimal is not.

The conformal invariance of Wintgen ideal property in the general case was pointed out in [6]. Thus it is appropriate to investigate and classify Wintgen ideal submanifolds under the framework of Möbius geometry. For this purpose, the submanifold theory in Möbius geometry established by Wang will be briefly reviewed in Section 2.

We will always assume that the Wintgen ideal submanifolds in consideration are not totally umbilic. Note that to have the shape operators taking the form in (1.2), the distribution \(\mathbb{D} = \text{Span}\{e_1, e_2\}\) is well defined. We call it \textit{the canonical distribution}. The first Möbius classification result was obtained by Li et al. [12].

\textbf{Theorem A (Li-Ma-Wang [12])}. Let \(x : M^m \to S^{n+p} \quad (m \geq 3)\) be a Wintgen ideal submanifold and it is not totally umbilic. If the canonical distribution \(\mathbb{D} = \text{Span}\{e_1, e_2\}\) is integrable, then locally \(x\) is Möbius equivalent to either one among the following three classes of examples in \(\mathbb{R}^{n+p}\):

(i) a cone over a minimal Wintgen ideal surface in \(S^{2+p}\);
(ii) a cylinder over a minimal Wintgen ideal surface in \(\mathbb{R}^{2+p}\);
(iii) a rotational submanifold over a minimal Wintgen ideal surface in \(\mathbb{H}^{2+p}\).

In this paper, we consider three-dimensional Wintgen ideal submanifolds \(x : M^3 \to S^5\), whose canonical distribution \(\mathbb{D}\) is not integrable. There is a Möbius invariant 1-form \(\omega\) associated with \(x\). For its definition as well as other basic equations and invariants, see Section 3.

Our main result is stated as below, which is proved in Section 4.

\textbf{Theorem B}. Suppose \(x : M^3 \to S^5\) is a Wintgen ideal submanifold whose canonical distribution \(\mathbb{D}\) is not integrable. It is Möbius equivalent to a minimal Wintgen ideal submanifold in a five-dimensional space form \(Q^5(c)\) if and only if the 1-form \(\omega\) is closed.

Under some further conditions, in Section 5 we characterize minimal Wintgen ideal submanifolds coming from Hopf bundle over complex curves in \(CP^2\). We also discuss the classification of Möbius homogeneous ones among Wintgen ideal 3-dimensional submanifolds in \(S^5\), which include the following example:

\[x : SO(3) \to S^5, \quad (u, v, u \times v) \mapsto \frac{1}{\sqrt{2}}(u, v).\]

As to the geometric meaning of the 1-form \(\omega\), we just mention that it could still be defined for Wintgen ideal submanifolds with dimension \(m \geq 4\). In a forthcoming paper [13], we will show that \(d\omega = 0\) is equivalent to the property that \(\mathbb{D} = \text{Span}\{e_1, e_2\}\) generates a 3-dimensional integrable distribution on