Stochastic Hamiltonian flows with singular coefficients

Dedicated to the 60th Birthday of Professor Michael Röckner

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Abstract In this paper, we study the following stochastic Hamiltonian system in $\mathbb{R}^2$ (a second order stochastic differential equation):

$$d\dot{X}_t = b(X_t, \dot{X}_t)dt + \sigma(X_t, \dot{X}_t)dW_t, \quad (X_0, \dot{X}_0) = (x, v) \in \mathbb{R}^2,$$

where $b(x, v): \mathbb{R}^2 \to \mathbb{R}^d$ and $\sigma(x, v): \mathbb{R}^2 \to \mathbb{R}^d \otimes \mathbb{R}^d$ are two Borel measurable functions. We show that if $\sigma$ is bounded and uniformly non-degenerate, and $b \in H^{2/3, 0}_p$ and $\nabla \sigma \in L^p$ for some $p > 2(2d+1)$, where $H^{2/3, 0}_p$ is the Bessel potential space with differentiability indices $\alpha$ in $x$ and $\beta$ in $v$, then the above stochastic equation admits a unique strong solution so that $(x, v) \mapsto Z_t(x, v) := (X_t, \dot{X}_t)(x, v)$ forms a stochastic homeomorphism flow, and $(x, v) \mapsto Z_t(x, v)$ is weakly differentiable with $\text{ess.sup}_{x, v} \mathbb{E}(\sup_{t \in [0, T]} |\nabla Z_t(x, v)|^q) < \infty$ for all $q > 1$ and $T > 0$. Moreover, we also show the uniqueness of probability measure-valued solutions for kinetic Fokker-Planck equations with rough coefficients by showing the well-posedness of the associated martingale problem and using the superposition principle established by Figalli (2008) and Trevisan (2016).

Keywords stochastic Hamiltonian system, weak differentiability, Krylov’s estimate, Zvonkin’s transformation, kinetic Fokker-Planck operator

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1 Introduction

Consider the following second order time-dependent stochastic differential equation (abbreviated as SDE):

$$d\dot{X}_t = b_t(X_t, \dot{X}_t)dt + \sigma_t(X_t, \dot{X}_t)dW_t, \quad (X_0, \dot{X}_0) = (x, v) \in \mathbb{R}^2,$$

where $b_t(x, v): \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}^d$ and $\sigma_t(x, v): \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}^d \otimes \mathbb{R}^d$ are two Borel measurable functions, $\dot{X}_t$ denotes the first order derivative of $X_t$ with respect to $t$, and $W_t$ is a $d$-dimensional standard Brownian motion on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$. When $\sigma = 0$, the above equation is the classical Newtonian mechanic equation, which describes the motion of a particle. When $\sigma \neq 0$, it
means that the motion is perturbed by some random external force. More backgrounds about the above stochastic Hamiltonian system are referred to [25, 28], etc.

It is noticed that if we let \( Z_t := (X_t, \dot{X}_t) \), then \( Z_t \) solves the following one order (degenerate) SDE:

\[
dZ_t = (\dot{X}_t, b(Z_t))dt + (0, \sigma_t(Z_t)dW_t), \quad Z_0 = z = (x, v) \in \mathbb{R}^{2d},
\]

(1.1)

whose time-dependent infinitesimal generator is given by

\[
\mathcal{L}^{a,b}_t f(x, v) := \text{tr}(a_t \cdot \nabla^2_x f)(x, v) + (v \cdot \nabla_x f)(x, v) + (b_t \cdot \nabla_v f)(x, v).
\]

(1.2)

Here, \( a_t(x, v) := \frac{1}{2}(\sigma_t \sigma_t^*) (x, v) \), \( \nabla^2_x f(x, v) \) stands for the Hessian matrix, the asterisk and \( \text{tr}(\cdot) \) denote the transpose and the trace of a matrix, respectively. Moreover, let \( \mu_t \) be the probability distributional measure of \( Z_t \) in \( \mathbb{R}^{2d} \). By Itô’s formula, one knows that \( \mu_t \) solves the following Fokker-Planck equation in the distributional sense:

\[
\partial_t \mu_t = (\mathcal{L}^{a,b}_t)^* \mu_t, \quad \mu_0 = \delta_z,
\]

(1.3)

where \( \delta_z \) is the Dirac measure at \( z \). More precisely, for any \( f \in C^2_c(\mathbb{R}^{2d}) \),

\[
\partial_t \mu_t(f) = \mu_t(\mathcal{L}^{a,b}_t f), \quad \mu_0(f) = f(z),
\]

where

\[
\mu_t(f) = \int f d\mu_t = Ef(Z_t).
\]

In the literature \( \mathcal{L}^{a,b}_t \) is also called kinetic Fokker-Planck or Kolmogorov’s operator.

During the past decade, there is an increasing interest in the study of SDEs with singular or rough coefficients. In the non-degenerate case, Krylov and Röckner [18] showed the strong uniqueness to the following SDE in \( \mathbb{R}^d \):

\[
dX_t = b_t(X_t)dt + dW_t, \quad X_0 = x,
\]

where \( b \in L^q_{\text{loc}}(\mathbb{R}^+; L^p(\mathbb{R}^d)) \) with \( \frac{3}{p} + \frac{2}{q} < 1 \). The argument in [18] is based on Girsanov’s theorem and some estimates from the theory of PDE. In this framework, Fedrizzi and Flandoli [11, 12] studied the well-posedness of stochastic transport equations with rough coefficients. When \( b \) is bounded measurable, the Malliavin differentiability of \( X_t \) with respect to sample path \( \omega \) and the weak differentiability of \( X_t \) with respect to starting point \( x \) were recently studied in [20] and [22], respectively. We also mention that weak uniqueness was studied in [1, 15] under rather weak assumptions on \( b \) (belonging to some Kato’s class). Moreover, the multiplicative noise case was studied in [32, 33, 35] by using Zvonkin’s transformation (see [36]) and some careful estimates of second order parabolic equations.

In the degenerate case, Chaudru de Raynal [7] firstly showed the strong well-posedness for SDE (1.1) under the assumptions that \( \sigma \) is Lipschitz continuous and \( b \) is \( \alpha \)-Hölder continuous in \( x \) and \( \beta \)-Hölder continuous in \( v \) with \( \alpha \in (\frac{2}{3}, 1) \) and \( \beta \in (0, 1) \). The proofs in [7] strongly depend on some explicit estimates for Kolmogorov operator with constant coefficients and Zvonkin’s transformation. In a recent joint work [30] with Wang, we also showed the strong uniqueness and homeomorphism property for (1.1) under weaker Hölder-Dini’s continuity assumption on \( b \). The proofs in [30] rely on a characterization of Hölder-Dini’s spaces and gradient estimates for the semigroup associated with the kinetic operator. Notice that in [7, 30], more general degenerate SDEs are considered, while, the case with critical differentiability indices \( \alpha = \frac{2}{3} \) and \( \beta = 0 \) is left open.

The purpose of this work is to establish a similar theory for degenerate SDE (1.1) as in Krylov and Röckner’s paper [18] (see also [35]). In particular, the critical indices \( \alpha = \frac{2}{3} \) and \( \beta = 0 \) are covered. More precisely, we aim to prove the following.

**Theorem 1.1.** Suppose that for some \( K \geq 1 \) and all \( (t, x, v) \in \mathbb{R}^+ \times \mathbb{R}^{2d} \),

\[
K^{-1} |\xi| \leq |\sigma_t^*(x, v)\xi| \leq K|\xi|, \quad \forall \xi \in \mathbb{R}^d,
\]

(UE)