Collection of problems proposed at 
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Variational Methods* 

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Abstract This collection of problems is based on the Problem Section held on May 24, 2007 during the International Conference on Variational Methods. These problems reflect various aspects of variational methods and are due to Professors Victor Bangert, Alain Chenciner, Ivar Ekeland, Nassif Ghoussoub, Zhaoli Liu, Paul Rabinowitz and Hans-Bert Rademacher. 

Keywords Variational method, nonlinear analysis, open problem 
MSC 58E05, 35A15, 34A34 

The International Conference on Variational Methods (ICVAM) was held at Nankai University during May 20–26, 2007. In the afternoon of May 24th, the organizing committee arranged a one-hour Problem Section in which many invited speakers proposed various open problems related to variational methods in different areas. The initial purpose of organizing such a Problem Section was to provide more opportunities for young students and researchers in the conference to understand the related subjects and to communicate with active mathematicians. 

This ‘Collection of Problems’ is based on the discussion in the Problem Section of May 24th and including also those which were submitted by Scientific Committee members and invited speakers of ICVAM after the conference. Problems listed in ‘Collection of Problems’ are arranged in alphabetical order of family names of proposers. 

The organizing committee would like to take this opportunity to thank all the people who made contributions to the success of the ICVAM and/or contributed to this ‘Collection of Problems’. 

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1 Problems proposed by Victor Bangert: Existence of integrable geodesic flows on surfaces of higher genus

The most interesting questions in the study of completely integrable geodesic flows concern the construction and investigation of explicit (classes of) examples of completely integrable geodesic flows. However, also the question if there are topological obstructions to the integrability of geodesic flows has found considerable interest, see e.g. Refs. [1.5–1.12]. All these results have in common that they only prove the non-existence of completely integrable geodesic flows with additional properties, e.g. with real analytic ($C^\omega$-) integrals. In particular, from Ref. [1.6] one knows that a compact surface of genus $> 1$ cannot carry a $C^\omega$-integrable geodesic flow. More recently and in contrast to these results, one has found examples of $C^\infty$-integrable geodesic flows on certain manifolds which do not admit $C^\omega$-integrable geodesic flows, see Refs. [1.1–1.4]. However, to my knowledge the question if such examples exist on compact surfaces of genus $> 1$ is open.

To make precise statements we fix the following notation. We let $M$ be a compact manifold with real analytic (=$C^\omega$) structure and we let $F: TM \to \mathbb{R}_{\geq 0}$ denote a Finsler metric on $M$. The unit tangent bundle of $F$ will be denoted by $T_1^1F$ and the geodesic flow on $T_1^1F$ by $\phi^F_t$.

**Definition 1.1** For $k \in \mathbb{N} \cup \{\infty, \omega\}$, a $C^k$-function $G$ defined in a neighborhood of $T_1^1F$ in $TM$ is called a $C^k$-integral of $\phi^F_t$ if $G \circ \phi^F_t = G|_{T_1^1F}$ for all $t \in \mathbb{R}$.

If one believes in the existence of $C^k$-integrable geodesic flows on surfaces of genus $> 1$ for some $k \leq \infty$, one should try to attack

**Problem 1.1** For some $2 \leq k \leq \infty$, find a compact surface $M$ of genus $> 1$ and a Finsler metric $F$ of class $C^k$ (outside the zero section) on $M$, for which there exists a $C^k$-integral $G$ of $\phi^F_t$, such that $G|_{T_1^1F}$ is regular on a dense (and open) subset of $T_1^1F$.

Since Riemannian metrics are much more rigid than Finsler metrics, it is conceivable that examples as in Problem 1.1 cannot exist in the Riemannian case. So, one could try to prove

**Problem 1.2** Let $g$ be a Riemannian metric of class $C^\infty$ on a compact surface $M$ of genus $> 1$. Then there does not exist a $C^\infty$-integral $G$ for the geodesic flow of $g$, such that $G|_{T_1^1g}$ is (Lebesgue) almost everywhere regular.

2 Problems proposed by Alain Chenciner: Extra symmetry of some minimizers

It happens that minimizers under symmetry constraints have more symmetry than a priori required. An example is the ‘figure eight solution’ of the Newtonian three-body problem with equal masses (Ref. [2.1]): It was originally obtained by minimizing the Lagrangian action among $T$-periodic loops