On g-s-supplemented subgroups of finite groups

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Abstract A subgroup H of a group G is said to be g-s-supplemented in G if there exists a subgroup K of G such that HK ⊴ G and H ∩ K ≤ H_sG, where H_sG is the largest s-permutable subgroup of G contained in H. By using this new concept, we establish some new criteria for a group G to be soluble.

Keywords Finite group, g-s-supplemented subgroup, Sylow subgroup, soluble group

MSC 20D10, 20D15, 20D20

1 Introduction

All groups in this paper are finite.

A subgroup H of a group G is said to be complemented in G if G has a subgroup K of G such that HK = G and H ∩ K = 1. A subgroup H of a group G is said to be supplemented in G if there exists a subgroup K of G such that HK = G. Obviously, a complemented subgroup is a special supplemented subgroup. It is well known that the supplemented subgroups play an important role in the study of finite groups. For example, Hall [6] proved that a group G is soluble if and only if every Sylow subgroup of G is complemented in G. Kegel [7,9] proved that a group G is soluble if every maximal subgroup of G has a cyclic supplement in G or if some nilpotent subgroup of G has a nilpotent supplement in G. Recently, some new special supplemented subgroups were introduced. For example, a group is said to be c-supplemented in G [12] if there exists a subgroup K of G such that

* Received November 7, 2008; accepted January 25, 2010

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\( HK = G \) and \( H \cap K \leq H_G \), where \( H_G \) is the largest normal subgroup of \( G \) contained in \( H \). For a formation \( \mathcal{F} \), a subgroup \( H \) of a group \( G \) is said to be \( \mathcal{F} \)-supplemented in \( G \) [4] if there exists a subgroup \( K \) such that \( HK = G \) and \( (H \cap K)H_G/H_G \) is \( \mathcal{F} \)-hypercentral in \( G/H_G \). A subgroup \( H \) is said to be weakly \( s \)-supplemented in \( G \) [11] if there exists a subgroup \( K \) of \( G \) such that \( HK = G \) and \( H \cap K \leq H_{sG} \), where \( H_{sG} \) is the largest \( s \)-permutable subgroup of \( G \) contained in \( H \) (note that a subgroup \( H \) of \( G \) is said to be \( s \)-permutable in \( G \) if \( HP = PH \) for any Sylow subgroup \( P \) of \( G \)). A subgroup \( H \) is said to be \( s \)-embedded in \( G \) if \( H \) has a normal subgroup \( K \) and an \( s \)-permutable subgroup \( C \) such that \( T \cap H \leq H_{sG} \) and \( HT = C \). By using the above special supplemented subgroups, people have obtained a series of interesting results (see Refs. [4,5,11,12]). Now, we consider a generalizer supplemented subgroup and give the following concept.

Definition 1.1 A subgroup \( H \) of a group \( G \) is said to be \( g \)-s-supplemented in \( G \) if there exists a subgroup \( K \) of \( G \) such that \( HK \leq G \) and \( H \cap K \leq H_{sG} \), where \( H_{sG} \) is the largest \( s \)-permutable subgroup of \( G \) contained in \( H \).

It is easy to see that every \( c \)-supplemented subgroup and every weakly \( s \)-supplemented subgroup is \( g \)-s-supplemented. However, the following examples show that the converse is not true.

Example 1.2 Let \( G = A \times B \), where \( A \) is a cyclic group of order 5 and \( B = \langle \alpha \rangle \), where \( \alpha \in \text{Aut}(A) \) with \( |\alpha| = 4 \). Since \(|G : \langle \alpha^2 \rangle A| = 2\), \( \langle \alpha^2 \rangle A \) is normal in \( G \). Then by \( \langle \alpha^2 \rangle \cap A = 1 \), we see that \( \langle \alpha^2 \rangle \) is \( g \)-s-supplemented in \( G \). However, \( \langle \alpha^2 \rangle \) is not weakly \( s \)-supplemented in \( G \). In fact, let \( \langle \alpha^2 \rangle \) be weakly \( s \)-supplemented in \( G \), and assume that \( K \) is a subgroup of \( G \) such that

\[
K \langle \alpha^2 \rangle = G, \quad K \cap \langle \alpha^2 \rangle \leq \langle \alpha^2 \rangle_{sG}.
\]

Then, clearly \( \langle \alpha^2 \rangle_{sG} = 1 \) or \( \langle \alpha^2 \rangle_{sG} = \langle \alpha^2 \rangle \). If \( \langle \alpha^2 \rangle_{sG} = 1 \), then \( \langle \alpha^2 \rangle \) is complemented in \( G \) and hence \( \langle \alpha^2 \rangle \) is complemented in \( B \), which contradicts the fact that \( B \) is a cyclic group. Assume that \( \langle \alpha^2 \rangle = \langle \alpha^2 \rangle_{sG} \). Then \( \langle \alpha^2 \rangle \) is an \( s \)-permutable subgroup of \( G \). If it follows that \( \langle \alpha^2 \rangle \leq O_2(G) \) (see the Lemma 2.2 below). This contradicts the fact that \( O_2(G) = 1 \). Therefore, \( \langle \alpha^2 \rangle \) is not weakly \( s \)-supplemented in \( G \), and consequently it is not \( c \)-supplemented in \( G \).

Example 1.3 Let \( G = A_4 = (Z_1 \times Z_2) \times Z_3 \) be the alternating group of degree 4. Since \( (Z_1 \cap Z_2) = 1 \) and \( (Z_1 \times Z_2) \) is normal in \( G \), \( Z_1 \) is \( g \)-s-supplemented in \( G \). But \( Z_1 \) is not weakly \( s \)-supplemented in \( G \) since \( G \) has no subgroup of order 6 and the supplemented subgroup of \( Z_1 \) in \( G \) is only \( G \).

All unexplained terminologies and notations are standard, the reader is referred to Refs. [3,10].

2 Preliminaries

Lemma 2.1 Let \( A, B \) and \( K \) be subgroups of a group \( G \).