Gradient estimates and Harnack inequalities for diffusion equation on Riemannian manifolds

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Abstract We derive the gradient estimates and Harnack inequalities for positive solutions of the diffusion equation $u_t = \Delta u^m$ on Riemannian manifolds. Then, we prove a Liouville type theorem.

Keywords Gradient estimate, Harnack inequality, diffusion equation, Riemannian manifold

MSC 53C21, 58J35

1 Introduction

Let $M$ be a complete Riemannian manifold, we consider the following diffusion equation

$$u_t = \Delta u^m$$

(1.1)

on $M \times [0, \infty)$, where $m$ is a positive constant. The equation is recently studied by many authors (e.g., [1,2,5]). When $m = 1$, (1.1) is the heat equation. Li and Yau [7] derived the gradient estimates and the Harnack inequalities for positive solutions of (1.1). Li [6] generalized their result to some nonlinear cases. In this paper, we consider the above diffusion equation ($m > 0$) on complete Riemannian manifolds. Following the discussion in [6], we prove the following gradient estimates and Harnack inequalities.

Theorem 1.1 Let $M$ be an $n$-dimensional complete Riemannian manifold with possibly empty boundary $\partial M$. Assume that $P \in M$, and $B_P(2R)$, the geodesic ball of radius $2R$ around $P$, does not intersect the boundary $\partial M$. We denote $-K(2R)$ to be the constant such that the Ricci curvature of $M$ is bounded from below by $-K(2R)$ in $B_P(2R)$, and $K(2R) \geq 0$. If $u(x,t)$ is a
positive solution of (1.1) on \( M \times [0, \infty) \) with \( A \leq u(x, t) \leq B \) in \( B_P(2R) \times [0, T] \) \((A, B \) are positive constants), then

1. in the case that \( m > 1, \forall x \in B_P(R), t \in [0, T] \), \( u(x, t) \) satisfies the estimate

\[
\frac{\| \nabla u \|^2}{u^2} - \frac{1}{\beta} \frac{u_t}{u} \leq \frac{1}{(m-1)\beta} \left( \frac{B}{A} \right)^{m-1} \frac{1}{t} + M_1(R)B^{(m-1)/2} + M_2(R)B^{m-1} + M_3(R)B^{2(m-1)},
\]

where

\[
M_1(R) = \frac{1}{\beta} \sqrt{\frac{m}{(m-1)\beta}} K(2R),
\]

\[
M_2(R) = \frac{m}{(m-1)\beta} \left( \frac{C_2}{R^2} + \frac{C_1(n-1)}{R} \right) \sqrt{K(2R)},
\]

\[
M_3(R) = \frac{C_1nm^4}{2(\beta + m)(m-1)R^2\beta^2}, \quad \beta = \frac{m}{4n} \left( \frac{A^{m-1}}{A^{m-1} + 1} \right),
\]

and \( C_1, C_2 \) are positive constants;

2. in the case that \( 0 < m < 1, \forall x \in B_P(R), t \in [0, T] \), \( u(x, t) \) satisfies the estimate

\[
\frac{\| \nabla u \|^2}{u^2} - \frac{1}{\beta} \frac{u_t}{u} \leq \frac{2}{(1-m)\beta} \left( \frac{B}{A} \right)^{1-m} \frac{1}{t} + M_4(R)A^{(m-1)/2} + M_5(R)A^{m-1} + M_6(R)A^{2(m-1)},
\]

where

\[
M_4(R) = \frac{1}{\beta} \sqrt{\frac{2mn}{(1-m)\beta}} K(2R),
\]

\[
M_5(R) = \frac{2m}{(1-m)\beta} \left( \frac{C_2}{R^2} + \frac{C_1(n-1)}{R} \right) \sqrt{K(2R)},
\]

\[
M_6(R) = \frac{C_1nm^4}{(1-m)(\beta + m)R^2\beta^2}, \quad \beta = -\frac{m^3(1-m)}{11n(B^{1-m} + 1)},
\]

and \( C_1, C_2 \) are positive constants.

As a corollary, we have the following global estimates.

**Theorem 1.2** Let \( M \) be an \( n \)-dimensional complete Riemannian manifold with \( \text{Ric}(M) \geq -K \) \((K \geq 0) \). If \( u(x, t): M \times (0, \infty) \rightarrow \mathbb{R}^+ \) is a positive solution of the diffusion equation (1.1) on \( M \) with \( A \leq u(x, t) \leq B \) \((A, B \) are positive constants), then

1. for \( m > 1 \), the following inequality holds:

\[
\frac{\| \nabla u \|^2}{u^2} - \frac{1}{\beta} \frac{u_t}{u} \leq \frac{1}{(m-1)\beta} \left( \frac{B}{A} \right)^{m-1} \frac{1}{t} + \frac{1}{\beta} \sqrt{\frac{nm}{(m-1)\beta}} B^{(m-1)/2} K.
\]