Nonexistence of Levi-degenerate hypersurfaces of constant signature in $\mathbb{CP}^n$

Alla Sargsyan

Abstract. Let $M$ be a smooth hypersurface of constant signature in $\mathbb{CP}^n$, $n \geq 3$. We prove the regularity for $\bar{\partial}_b$ on $M$ in bidegree $(0, 1)$. As a consequence, we show that there exists no smooth hypersurface in $\mathbb{CP}^n$, $n \geq 3$, whose Levi form has at least two zero-eigenvalues.

1. Introduction

The question of nonexistence of smooth Levi-degenerate hypersurfaces in $\mathbb{CP}^n$ has attracted a great interest in recent years and takes its origin in the foliation theory, [4] and [9]. Several nonexistence results were obtained for Levi-flat CR manifolds. Recall that a CR manifold $M$ is called Levi flat, if there exists a local foliation of $M$ by complex manifolds, whose dimension coincides with the CR dimension of $M$. In the hypersurface case this is equivalent to vanishing of the Levi form of $M$. The nonexistence of real-analytic Levi-flat hypersurfaces in projective spaces for $n \geq 3$ was discussed by Lins Neto in [9]. Siu has shown that there does not exist any $C^8$-smooth Levi-flat hypersurface in $\mathbb{CP}^n$ for $n \geq 2$, [12] and [13]. The regularity assumption on the hypersurface was relaxed to $C^4$ by Iordan for $n \geq 2$, [8]. The conjecture of Siu on the nonexistence of higher codimensional smooth Levi-flat CR manifolds in compact symmetric spaces was proved by Brinkschulte in [2]. Cao and Shaw have proved that there does not exist any Lipschitz Levi-flat hypersurface in $\mathbb{CP}^n$ for $n \geq 3$, [3]. The case $n=2$ is still open.

We show that in the more general case of Levi-degenerate manifolds (whose Levi form does not necessarily vanish) the nonexistence depends essentially on the signature of the hypersurface. In our paper we use Siu’s idea to reduce the problem to the regularity of the tangential Cauchy–Riemann operator $\bar{\partial}_b$ in bidegree $(0, 1)$. One way to prove the regularity is to derive this from $L^2$-weighted estimates using the Bochner–Kodaira–Nakano inequality. However there are some obstructions to
prove the desired estimates in terms of the usual Fubini–Study metric of $\mathbb{CP}^n$. Following Brinkschulte’s arguments [2] we construct a new metric in $\mathbb{CP}^n \setminus M$, which provides good estimates for the curvature term in the Bochner–Kodaira–Nakano inequality at points near to the hypersurface. To extend the obtained estimates over the whole projective space we use Ohsawa’s method of pseudo-Runge pairs.

**Theorem 1.1.** Let $M$ be a smooth real hypersurface in $\mathbb{CP}^n$, $n \geq 3$, having a constant signature $(q^-, q^0, q^+)$ with $q^0 + \min\{q^-, q^+\} \geq 2$ and $q^0 \geq 1$.

If $f \in C^\infty_0(M) \cap \text{Ker} \partial_b$, then for every $k \in \mathbb{N}$ there exists $u \in C^k(M)$ such that $\partial_b u = f$ on $M$.

The above regularity implies our main nonexistence result.

**Theorem 1.2.** There exists no smooth real hypersurface in $\mathbb{CP}^n$, $n \geq 3$, whose Levi form has constant signature and satisfies one of the following conditions:

(i) the Levi form has at least two zero eigenvalues;
(ii) the Levi form has at least one zero eigenvalue and two eigenvalues of opposite signs.

Acknowledgements. The results of this paper are based on the investigations of Dr. Judith Brinkschulte and I would like to express my deep gratitude to her for the guidance throughout the research.

### 2. Construction of the new metrics

We consider the complex projective space $\mathbb{CP}^n$, $n \geq 3$, with its standard Fubini–Study metric $\omega_{FS}$. Let $M$ be a smooth closed real hypersurface in $\mathbb{CP}^n$ represented as

$$M = \{ z \in U : \rho(z) = 0 \},$$

where $\rho : U \to \mathbb{R}$ is a smooth function in an open neighborhood $U \subset \mathbb{CP}^n$ of $M$, and $d\rho(z) \neq 0$ on $M$. The hypersurface $M$ divides $\mathbb{CP}^n$ into two sets $\Omega^+$ and $\Omega^-$ with $\Omega^- \cap U = \{ z \in U : \rho(z) < 0 \}$ and $\Omega^+ \cap U = \{ z \in U : \rho(z) > 0 \}$. We denote by $(q^-, q^0, q^+)$ the signature of $M$, that means that the Levi form $L_z(\rho)$ of $\rho$ has exactly $q^-$ negative, $q^0$ zero and $q^+$ positive eigenvalues on $T_z^{1,0} M$ at each point $z \in M$. Clearly, $q^- + q^0 + q^+ = n - 1$. Next, we denote by $\delta_M(z)$ the Fubini distance from a point $z \in \mathbb{CP}^n$ to the hypersurface $M$ and by $K$ a compact set in $\mathbb{CP}^n$ that contains all the points near which $\delta_M(z)$ fails to belong to $C^2$.

It was proved by Matsumoto [10], that if $\Omega$ is a weakly $q$-convex set in $\mathbb{CP}^n$, $1 \leq q \leq n$, then the Levi form of $-\log \delta_M$ has at least $q+1$ positive eigenvalues in