Verdier specialization via weak factorization

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Abstract. Let \( X \subset V \) be a closed embedding, with \( V \setminus X \) nonsingular. We define a constructible function \( \psi_{X,V} \) on \( X \), agreeing with Verdier’s specialization of the constant function \( 1_V \) when \( X \) is the zero-locus of a function on \( V \). Our definition is given in terms of an embedded resolution of \( X \); the independence of the choice of resolution is obtained as a consequence of the weak factorization theorem of Abramovich–Karu–Matsuki–Włodarczyk. The main property of \( \psi_{X,V} \) is a compatibility with the specialization of the Chern class of the complement \( V \setminus X \). With the definition adopted here, this is an easy consequence of standard intersection theory. It recovers Verdier’s result when \( X \) is the zero-locus of a function on \( V \).

Our definition has a straightforward counterpart \( \Psi_{X,V} \) in a motivic group. The function \( \psi_{X,V} \) and the corresponding Chern class \( c_{SM}(\psi_{X,V}) \) and motivic aspect \( \Psi_{X,V} \) all have natural ‘monodromy’ decompositions, for any \( X \subset V \) as above.

The definition also yields an expression for Kai Behrend’s constructible function when applied to (the singularity subscheme of) the zero-locus of a function on \( V \).

1. Introduction

Consider a family \( \pi: V \to D \) over the open disk, satisfying a suitable condition of local triviality over \( D \setminus \{0\} \). In [33], J.-L. Verdier defines a ‘specialization morphism’ for constructible functions, producing a function \( \sigma_*(\varphi) \) on the central fiber \( X \) of the family for every constructible function \( \varphi \) on \( V \). The key property of this specialization morphism is that it commutes with the construction of Chern classes of constructible functions in the sense of MacPherson [23]; cf. Theorem 5.1 in Verdier’s note. The specialization morphism for constructible functions is induced from a morphism at the level of constructible sheaves \( \mathcal{F} \), by taking alternating sums of ranks for the corresponding complex of nearby cycles \( R\Psi_\pi\mathcal{F} \).

The main purpose of this note is to give a more direct description of the specialization morphism (in the algebraic category, over algebraically closed fields of characteristic 0), purely in terms of constructible functions and of resolution of singularities, including an elementary proof of the basic compatibility relation with
Chern classes. We will assume that $V$ is nonsingular away from $X$, and focus on the case of the specialization of constant functions; by linearity and functoriality properties, this suffices in order to determine $\sigma_*$ in the situation considered by Verdier.

On the other hand, the situation we consider is more general than the specialization template recalled above: we define a constructible function $\psi_{X,V}$ for every proper closed subscheme $X$ of a variety $V$ (such that $V \setminus X$ is nonsingular), which agrees with Verdier’s specialization of the constant function $1_V$ when $X$ is the fiber of a morphism from $V$ to a nonsingular curve.

The definition of $\psi_{X,V}$ (Definition 2.1) is straightforward, and can be summarized as follows. Let $w: W \to V$ be a proper birational morphism such that $W$ is nonsingular, and $D=w^{-1}(X)$ is a divisor with normal crossings and nonsingular components, and for which $w|_{W\setminus D}$ is an isomorphism. Then define $\psi_{D,W}(p)$ to be $m$ if $p$ is on a single component of $D$ of multiplicity $m$, and 0 otherwise; and let $\psi_{X,V}$ be the push-forward of $\psi_{D,W}$ to $X$.

Readers who are familiar with Verdier’s paper [33] should recognize that this construction is implicit in Section 5 of that paper, if $X$ is the zero-locus of a function on $V$. Our contribution is limited to the realization that the weak factorization theorem of [1] may be used to adopt this prescription as a definition, that the properties of this function follow directly from the standard apparatus of intersection theory, and that this approach extends the theory beyond the specialization situation considered by Verdier (at least in the algebraic case). Denoting by $c_{SM}(\cdot)$ the Chern–Schwartz–MacPherson class of a constructible function, we prove the following theorem.

**Theorem I.** Let $i: X \to V$ be an effective Cartier divisor. Then

$$c_{SM}(\psi_{X,V}) = i^*c_{SM}(1_{V\setminus X}).$$

An expression for $i^*c_{SM}(1_{V\setminus X})$ in terms of the basic ingredients needed to define $\psi_{X,V}$ as above may be given as soon as $i: X \to V$ is a regular embedding (cf. Remark 3.5). In fact, with suitable positions, Theorem I holds for arbitrary closed embeddings $X \subset V$ (Theorem 3.3).

Theorem I reproduces Verdier’s result when $X$ is a fiber of a morphism from $V$ to a nonsingular curve; in that case (but not in general) $1_{V\setminus X}$ may be replaced with $1_V$, as in Verdier’s note. The definition of $\psi_{X,V}$ is clearly compatible with smooth maps, and in particular the value of $\psi_{X,V}$ at a point $p$ may be computed after restricting to an open neighborhood of $p$. Thus, Verdier’s formula for the specialization function in terms of the Euler characteristic of the intersection of a nearby fiber with a ball (Section 4 in [33]) may be used to compute $\psi_{X,V}$ if $X$ is a divisor in $V$, over $\mathbb{C}$. 