Abstract The main results of this paper give two criteria for certain infinite series of rational numbers to be Liouville. Some examples are also included.

Keywords Liouville numbers · Infinite series

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1 Introduction

Let \( \alpha \) be a real number. If for every positive real number \( r \) there exist integers \( p \) and \( q \) such that \( 0 < |\alpha - \frac{p}{q}| < \frac{1}{qr} \) then the number \( \alpha \) is called Liouville.

A survey of results concerning Liouville numbers and using Mahler method can be found in the book of Nishioka [4]. Also the results of Petruska [5] establish several interesting criteria concerning strong Liouville numbers. The latter was first defined by Erdős in [1].

It is relatively easy to prove that the number \( \sum_{n=1}^{\infty} \frac{1}{n!} \) is Liouville. This suggests similar results for other infinite series. In 1975 Erdős [2] proved a criterion for Liouville numbers.

Theorem 1.1 (Erdős) For strictly increasing sequence \( \{a_n\}_{n=1}^{\infty} \) suppose that

\[
\limsup_{n \to \infty} a_n^{1/n} = \infty
\]

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for every $t > 0$ and that there exists $\varepsilon_0 > 0$ and $n_0(\varepsilon_0) \in \mathbb{N}$ such that
$$a_n > n^{1+\varepsilon_0}$$
if $n > n_0(\varepsilon_0)$. Then
$$\alpha = \sum_{n=1}^{\infty} \frac{1}{a_n}$$
is a Liouville number.

Recently in [3] Hančl introduced the concept of Liouville sequences in the following way.

**Definition 1.1** We say a sequence of positive real numbers $\{a_n\}_{n=1}^{\infty}$ is Liouville if for every sequence of positive integers $\{c_n\}_{n=1}^{\infty}$ the sum $\sum_{n=1}^{\infty} \frac{1}{a_n c_n}$ is a Liouville number.

In [3], improving on Theorem 1.1, two criteria for a sequence to be Liouville are given.

### 2 Main results

**Theorem 2.1** Let $\varepsilon$ be a positive real number. Let $\{a_k\}_{k=1}^{\infty}$ and $\{b_k\}_{k=1}^{\infty}$ be two sequences of positive integers such that
$$\limsup_{k \to \infty} \frac{\log a_{k+1}}{\log (a_1 a_2 \cdots a_k)} = \infty.$$ (2.1)

Suppose for every sufficiently large $k$ that
$$\left(1+\varepsilon\right) \sqrt{\frac{a_{k+1}}{b_{k+1}}} \geq \sqrt{\frac{a_k}{b_k}} + 1.$$ (2.2)

Then the number
$$\xi = \sum_{k=1}^{\infty} \frac{b_k}{a_k}$$
is Liouville.

**Example 2.1** Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive integers such that $a_1 = 1$ and that
$$a_{n+1} = \begin{cases} (a_1 a_2 \cdots a_n)^{n+1}, & \text{if } 10 \mid n \\ [a_n + 2\sqrt{a_n} + 2], & \text{otherwise.} \end{cases} (n \in \mathbb{N})$$

As an immediate consequence of Theorem 2.1 we obtain that the number
$$\sum_{n=1}^{\infty} \frac{2\left[\frac{n-1}{10}\right] - \left[\frac{n}{10}\right]}{a_n}$$
is Liouville.