A simpler and tighter redundant Klee–Minty construction

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Abstract By introducing redundant Klee–Minty examples, we have previously shown that the central path can be bent along the edges of the Klee–Minty cubes, thus having $2^n - 2$ sharp turns in dimension $n$. In those constructions the redundant hyperplanes were placed parallel with the facets active at the optimal solution. In this paper we present a simpler and more powerful construction, where the redundant constraints are parallel with the coordinate-planes. An important consequence of this new construction is that one of the sets of redundant hyperplanes is touching the feasible region, and $N$, the total number of the redundant hyperplanes is reduced by a factor of $n^2$, further tightening the gap between iteration-complexity upper and lower bounds.

Keywords Linear programming · Klee–Minty example · Interior point methods · Worst-case iteration complexity · Central path

1 Introduction

Introduced by Dantzig [1] in 1947, the simplex method is a powerful algorithm for linear optimization problems. In 1972 Klee and Minty [7] showed that the simplex method may take an exponential number of iterations. More precisely, they presented an optimization problem over an $n$-dimensional squashed cube, and proved that a variant of the simplex method visits all of its $2^n$ vertices. The actual pivot rule in [7] was the most negative reduced cost pivot rule that is frequently referred to as “Dantzig rule”.

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Thus, its iteration-complexity is exponential in the worst case, as $2^n - 1$ iterations are needed for solving this $n$-dimensional linear optimization problem. Variants of the Klee–Minty $n$-cube have been used to prove exponential running time for most pivot rules, see [11] and the references therein for details. Stimulated mostly by the Klee–Minty worst-case example, the search for a polynomial algorithm for solving linear optimization problems has been started. In 1979, Khachiyan [6] introduced the polynomial ellipsoid method for linear optimization problems. In spite of its polynomial complexity bound, the ellipsoid method turned out to be inefficient in computational practice.

In 1984 in his seminal work, Karmarkar [5] proposed a more efficient polynomial time algorithm that sparked the research on polynomial interior point methods (IPMs). Unlike the simplex method which goes along the edges of the polyhedron corresponding to the feasible region, interior point methods pass through the interior of this polyhedron. Starting at the analytic center, most IPMs follow the so-called central path and converge to the analytic center of the optimal face; see e.g., [9,14]. It is well known that the number of iterations needed to have the duality gap smaller than $\epsilon$ is upper-bounded by $O(\sqrt{N \ln \frac{v_0}{\epsilon}})$, where $N$ and $v_0$ respectively denote the number of inequalities and the duality gap at the starting point. Then, the standard rounding procedure [9] can be used to compute an exact optimal solution.

In 2004, Deza et al. [2] showed that, the central path of a redundant representation of the Klee–Minty $n$-cube may trace the path followed by the simplex method. More precisely, an exponential number of redundant constraints parallel to the facets passing through the optimal vertex are added to the Klee–Minty $n$-cube to force the central path to visit an arbitrary small neighborhood of all the vertices of that cube, thus having $2^n - 2$ sharp turns. In this construction, uniform distances for the redundant constraints have been chosen and consequently the number of the inequalities for the highly redundant Klee–Minty $n$-cube becomes $N = O(n^2 2^{6n})$, which is further improved to $N = O(n 2^{3n})$ in [4] by a meticulous analysis. In [3], by allowing the distances of the redundant constraints to the corresponding facets to decay geometrically, the number of the inequalities $N$ is significantly reduced to $O(n^{3/2} 2^{2n})$. As shown in [3], a standard rounding procedure can be employed after $O(\sqrt{Nn})$ iterations that gives the optimal solution. This result also underlines that the reduction of the number of the redundant inequalities will further tighten the iteration-complexity lower and upper bounds.

In this paper, we simplify the construction in [3] by putting the redundant constraints parallel to the coordinate hyperplanes at geometrically decaying distances, and show that fewer redundant inequalities, only in the order of $N = O(n 2^{2n})$, is needed to bend the central path along the edges of the Klee–Minty $n$-cube. This yields a tighter iteration-complexity upper bound $O(n^{1/2} 2^n)$ while retaining the same lower bound $\Omega(2^n)$. In other words, the number of iterations is bounded below by $\Omega(\sqrt{\frac{N}{\ln N}})$ and above by $O(\sqrt{N \ln N})$.

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1 See p. 407 for the definition of the analytic center and the central path in case of redundant Klee–Minty cubes.