Imaging Analysis by Means of Fractional Fourier Transform

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Abstract Starting from the diffraction imaging process, we have discussed the relationship between optical imaging system and fractional Fourier transform, and proposed a specific system which can form an inverse amplified image of input function.

Key words imaging analysis, fractional Fourier transform
PACS 42.30

1 Optical Imaging and Fractional Fourier Transform

An imaging problem of an optical system can be generalized into a process in which the main purpose is to find the location of output image (or image plane) where the complex amplitude distribution is similar to distribution on the input plane (or object plane). When the phase distribution is not of interest, the image can be fully described by the intensity distribution.

Usually, the intensity distribution can be obtained with geometrical methods in which diffraction effect is negligible. However it must be remembered that only considering diffraction, can lead to a full and complete understanding of optical systems, as well as that of imaging processes.

Fourier optics theory is based on Huggens Fresnel principle, which relates two complex amplitude distribution on two planes. If the system between the input and output planes is specifically fixed in such a way that the Fourier transform is achieved, then the complex amplitude distribution \( u(x_0, y_0) \) at the input plane and \( u(x, y) \) at the output plane are connected by

\[
\mathcal{F}_1[u_0(x_0, y_0)] = u(x, y) = \int u_0(x_0, y_0) \exp[-j \frac{2\pi}{\lambda f} (xx_0 + yy_0)] dx_0 dy_0. \tag{1}
\]

In 1993, Mendlovic and Ozaktas first introduced the concept of fractional Fourier transform (FRT) into optics\(^1\). Theoretically, it is reasonable that there exist much more optical systems which can perform FRT than systems which can perform Fourier transform. The prediction turns out to be true. Many systems for implementing FRT have been proposed\(^2-5\), each having infinite transform of different order. Furthermore, because of additivity of fractional order and the commutative law, the number of configurations of transform systems is infinite.

To advance our discussion, we present the integral form of FRT here.

\[
\mathcal{F}_1[u_0(x_1, y_1)] = u(x, y) = \int u_0(x_1, y_1) \exp[-j \frac{2\pi}{\lambda F_1 \sin \phi} (xx_1 + yy_1) + j \frac{\pi}{2} \left( x^2 + x_1^2 + y^2 + y_1^2 \right)] dx_1 dy_1. \tag{2}
\]

where \( F_1 \) is a scaling factor determined by the sys-
tem, \( \lambda \) is wavelength, \( \phi \) is linked to fractional order \( p \) by \( \phi = \frac{\pi}{2} p \). From this definition, we can obtain,

\[
u_2(x, y) = u_0(-x_1, -y_1), \quad |\nu_2(x, y)|^2 = |u_0(-x_1, -y_1)|^2.
\]

This equation means that the object function is rotated by \( \pi \), the resulting image is the inverse of the original, as shown in Fig. 2. The \( x_1, y_1 \) and \( xy \) planes are respectively the input and output planes of a system which may include several subsystems.

![Fig. 2 An optical system capable of performing 2th FRT](image)

Now, if we have a system consisting of two cascaded subsystems, each has the capacity of performing FRT with order \( p_1 \) or \( p_2 (p_1 + p_2 = 2) \), and in one of which the coordinates decrease or increase by a factor of \( \beta \) after transform of this subsystem, the overall transform effect of the system can be expressed with the following equation.

\[
u(x, y) = u_0(-\beta x_1, -\beta y_1).
\] (3)

Physically, the above equation means that the input object function is inverted and amplified by a factor of \( \beta \), that is \( |\nu(x, y)|^2 = |u_0(-\beta x_1, -\beta y_1)|^2 \).

### 2 Imaging Analysis of a Specific System in Terms of Fractional Fourier Transform

Fig. 3 shows such a sample system. In the system, lens 1 and lens 2 form a subsystem illuminated by convergent wave, and is assumed to perform fractional Fourier transform of order \( p_1 \). Lens 3 with two symmetrical free distance \( z_2 \) is another subsystem which is assumed to implement fractional Fourier of transform of order \( p_2 \). After derivation, we can arrive at

\[
u(x^{'}, y^{'}) = c_1 \int u_0(x_1, y_1) \exp\left[\frac{j\pi}{\lambda} \left( x_1^2 + y_1^2 \right) \right] \exp\left[\frac{j\pi}{\lambda A} (x^2 + y^2) \right] dx_1 dy_1,
\] (4)

where \( A = \frac{z_2 f_s}{f_b - z_1} \). If we let \( x^{'}, y^{'}, x, y \) are real numbers, the input function \( u_0(x_1, y_1) \) will take a new form \( u_0^{'}(x_1, y_1) \) in the new coordinate scale of \((x_1, y_1) \) which is meant to be the same as the coordinate scale of \((x^{'}, y^{'}) \) in the FRT, while it is obvious that \( u_0^{'}(x_1, y_1) = u_0(\beta x_1, \beta y_1) \).

\[
u(x^{'}, y^{'}) = c_1 \int u_0^{'}(x_1, y_1) \exp\left[\frac{j\pi}{\lambda} \left( x_1^2 + y_1^2 \right) \right] \exp\left[\frac{j\pi}{\lambda A} (x^2 + y^2) \right] dx_1 dy_1.
\] (5)

Now, let \( f_a = \beta A = F_1 \tan \phi_1, \beta \beta_1 = F_1 \sin \phi_1 \) and \( \phi_1 = \frac{p_1}{2} \), then

\[
u(x^{'}, y^{'}) = \frac{F_1 \sin \phi_1}{\beta} \int u_0^{'}(x_1, y_1) \exp\left[\frac{j\pi}{\lambda F_1 \tan \phi_1} \left( x_1^2 + y_1^2 \right) \right] \exp\left[\frac{-j2\pi}{\lambda F_1 \sin \phi_1} (x^2 + y^2) \right] dx_1 dy_1.
\] (6)

In addition, for the second subsystem consisting of two free space with distance \( z_2 \) and lens with focal length \( f_c \), we have \( z_2 = \frac{F_1 (1 - \cos \phi_2)}{\sin \phi_2}, f_c = \frac{F_1}{\sin \phi_2} \), \( \phi_2 = \frac{p_2}{2} \), and \( p_2 = 2 - p_1 \). So we get

\[
u(x, y) = c_2 \int u(x^{'}, y^{'}) \exp\left[\frac{j\pi}{\lambda F_1 \tan \phi_2} \left( x^2 + y^2 \right) \right] \exp\left[\frac{-j2\pi}{\lambda F_1 \sin \phi_2} (x^{'2} + y^{'2}) \right] dx^{'} dy^{'}.
\] (7)

The expression Eq. (6) and Eq. (7) can be written as

\[
u(x^{'}, y^{'}) = F^{(p_1)} \{ u_0(x_1, y_1) \},
\] (8)

\[
u(x, y) = F^{(p_2)} \{ u(x^{'}, y^{'}) \}.
\] (9)

From Eq. (8) and Eq. (9), we get

\[
u(x, y) = F^{(p_2)} \{ F^{(p_1)} \{ u_0(x_1, y_1) \} \}.
\]