Numerical Evaluation of CPV Boundary Integrals with Symmetrical Quadrature Schemes

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Abstract Stemming from the definition of the Cauchy principal values (CPV) integrals, a newly developed symmetrical quadrature scheme was proposed in the paper for the accurate numerical evaluation of the singular boundary integrals in the sense of CPV encountered in the boundary element method. In the case of inner-element singularities, the CPV integrals could be evaluated in a straightforward way by dividing the element into the symmetrical part and the remainder(s). And in the case of end-singularities, the CPV integrals could be evaluated simply by taking a tangential distance transformation of the integrand after cutting out a symmetrical tiny zone around the singular point. In both cases, the operations are no longer necessary before the numerical implementation, which involves the dull routine work to separate out singularities from the integral kernels. Numerical examples were presented for both the two- and the three-dimensional boundary integrals in elasticity. Comparing the numerical results with those by other approaches demonstrates the feasibility and the effectiveness of the proposed scheme.

Key words boundary element method (BEM), singular boundary integral, symmetrical integration, distance transformation, numerical evaluation.

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1 Introduction

The boundary integral equations arise in many problems of physics and engineering. One of the primary mathematical and numerical difficulties of boundary integral equation appears to be the treatment of the singular kernels. There are three different types of singularities in the general boundary integral formulations, namely the weak singularity, Cauchy principal value (CPV) singularity and hypersingularity. Much research work has been performed and many numerical techniques have been proposed to cope with the singular kernels in the past few years[1-14]. These techniques can be summarized roughly into three categories: the local regularization techniques[1-3], the integral transformation techniques of various kinds[4-8], and the global regularization techniques[8-14]. The choice of the techniques is a matter of problem dependence or of personal-preference dependence because there are both merits and shortcomings in either of them.

The local regularization techniques[1-3] afford us a systematical way to separate out singularities from the integral kernels step by step, which involves the dull routine work, as the expressions of the kernels are generally complicated. The resultant kernels become non-singular after the separation and then some accurate integration of the function can be added back to the integral[3]. The integral thus evaluated does not contain the corner information if the source point is placed on the corner. In the use of transformation techniques[7,8], it is often the case that some skills or experiences are required by choosing appropriate parameters to fit the nature of the kernels. And the method of treating the hypersingularity with the transformation techniques seems not to be available yet. The most famous of the global regularization techniques is the well-known rigid body motion, for example in elasticity, by subtracting a uniform displacement field from the global displacement boundary integral equation if the continuation condition is satisfied. In addition to this, the linear state tractions and the displacements associated with a state of constant stress in the whole body can be used to regularize the global stress boundary integral equation[10-12]. The
global regularization techniques, although having got rid of the work of corner treatment, make the computer programming somewhat complicated as the global nature. And it is clear that the same accuracy level of numerical results in the traction formulation has not been achieved yet as that of the displacement formulation.\textsuperscript{15}

The present work is focused on the local numerical treatment, which is considered as the simplest among those ever existing, of CPV boundary integrals encountered in boundary element method. This is partially because with the shape function manipulation the hypersingular integral can be decomposed into a CPV integral and a hypersingular integral with naked kernels. The latter can be evaluated over straight lines in 2-D cases or cone surfaces in 3-D cases\textsuperscript{15} by using the divergence-free properties of the hypersingular kernels. Two symmetrical integration schemes are proposed for the case of inner-element singularities and the case of the end-singularities, respectively. Numerical tests are performed and compared with other techniques for both two- and three-dimensional boundary integrals in elasticity, showing the feasibility and the effectiveness of the proposed scheme.

2 Representation of CPV Integrals

We start from the well-known two boundary integral equations, referred to as the displacement and the stress boundary integral equations, respectively, in elasticity without considering body force, in a domain \( \Omega \) with boundary \( \Gamma \), written in terms of the boundary tractions \( \tau_j \) and boundary displacements \( u_j \), as follows:

\[\begin{align*}
\sum_{i}^{n} \int_{\Gamma} \tau_i(x) u_{ij}^*(x,y) \hat{u}_i(y) d\Gamma(x) - \int_{\Delta} \int_{\Delta'} \tau_i(x,y) d\Gamma(x) - \text{CPV} \int_{\Delta} \int_{\Delta'} \tau_i^*(x,y) d\Gamma(x) & = \text{HFP} \int_{\Delta} \int_{\Delta'} \tau_i^*(x,y) d\Gamma(x) \\
\int_{\Gamma} \int_{\Delta} \hat{u}_j(y) \tau_j(y) d\Gamma(y) & = \text{HFP} \int_{\Delta} \int_{\Delta'} \tau_j(y) d\Gamma(y) 
\end{align*}\] (1)

where \( \sigma_{ij} \) stands for the stress; \( x \) and \( y \) represent the field and the source points, respectively; the kernels \( u_{ij}^* \) and \( \tau_{ij}^* \) are Kelvin's solutions; \( u_{jk}^* \) and \( \tau_{jk}^* \) denote the derived kernels from Kelvin's solutions; \( \gamma \) is the coefficient of free terms; the term HFP means that the boundary integrals are hypersingular in the sense of Hadamard finite part; \( \Delta' \) stands for the part of boundary that will be divided into one element resulting in the inner-element singularities, or two elements resulting in the end-singularities. The three CPV integrals, which have singularity orders of \( O\left(r^{-1}\right) \) and \( O\left(r^{-2}\right) \) in two- and three-dimensions, respectively, in the above two equations are the main concerns of the present paper, where \( r \) denotes the Cartesian distance between \( x \) and \( y \).

For simplicity, let \( K(x, y) \) represent one of the CPV singular kernels. Firstly we consider the two-dimensional case where the singular kernel can be written in the form of local coordinate system as follows:

\[ K(x, y) = \frac{1}{r} K^0(x, y) = \frac{1}{(\xi - c)r^*(\xi)} K^0(\xi) \] (3)

where \( c \) stands for the local coordinate of \( y \). The CPV singular integrals over a line boundary element can be written in a unified form as

\[ I_{CPV} = \text{CPV} \int_{\Delta} \frac{1}{(\xi - c)} K^0(\xi) G(\xi) d\xi = \text{CPV} \int_{\Delta} \frac{1}{\xi - c} f(\xi) d\xi, c \in (a, b) \] (4)

where \( f(\xi) \) represents the regular function, \( i.e., \) the product of the shape function \( \phi \), the Jacobian \( G \) and the regular parts of the kernels. Without losing generality, suppose \( c \geq 0.5(a + b) \) as shown in Fig. 1, decompose the integration span of the above integral into a symmetrical part and a remainder, and rewrite the CPV integrals as

\[ I_{CPV} = \text{CPV} \int_{\Delta} \left( \frac{1}{\xi - c} f(\xi) d\xi + \frac{1}{\xi - c} f(\xi) d\xi \right) \]

\[ = I_{SYM} + I_{REM} \] (5)

Fig. 1 Symmetrical quadrature scheme over a line boundary element in the local coordinate system in two dimensions

As the remainder integral \( I_{REM} \) is a regular one, we need only consider the symmetrical part. Taking a linear transformation of \( \rho = \xi - c \) to shift the singular point to the center position of the symmetrical integration span, we have