General Exact Penalty Functions in Integer Programming

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Abstract In this paper, the general exact penalty functions in integer programming were studied. The conditions which ensure the exact penalty property for the general penalty function with one penalty parameter were given and a general penalty function with two parameters was proposed.

Key words integer programming, exact penalty function, penalty parameter.

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1 Introduction

We consider the integer programming problem

\[
(P) \quad \min f(x) \\
\text{s.t.} \quad g_i(x) \leq 0, \quad i = 1, \ldots, m, \quad x \in X_I,
\]

where \(X_I\) is a finite integer set. Denote the feasible set \(\{x \in X_I \mid g_i(x) \leq 0, \ i = 1, \ldots, m\}\) by \(S_I\), and the set of optimal solutions of \((P)\) by \(G(P)\). We assume that \(S_I \neq \emptyset\) and \(S_I / G(P) \neq \emptyset\). Without loss of generality, assume that \(f(x) > 0, \ \forall x \in X_I\).

Exact penalty function methods in integer programming attempt to solve \((P)\) by solving an unconstrained integer programming problem. Sinclair\(^1\) showed that any optimal solution of following unconstrained integer programming problem with sufficiently large \(\rho\) is an optimal solution of \((P)\):

\[
\min f(x) + \rho L(x) \\
\text{s.t.} \quad x \in X_I,
\]

where \(L : \mathbb{R}^n \rightarrow \mathbb{R}\) satisfies

\[
L(x) = \begin{cases} 
0 & \text{if } x \in X_I \setminus S_I, \\
\geqslant \rho \sum_{i=1}^{m} g_i(x) & \text{if } x \in S_I.
\end{cases}
\]

Sun and Li\(^2\) proposed a logarithmic-exponential penalty term function

\[
\frac{1}{\rho} \ln \left(1 + \sum_{i=1}^{m} \exp(p g_i(x))\right)
\]

by approximating \(P(x) = \max \{0, g_1(x), \ldots, g_m(x)\}\). Obviously \(P(x)\) is a special case of \(L(x)\). Note that there are two penalty parameters \(\rho\) and \(p\) in the exact penalty function

\[
f(x) + \rho \frac{1}{\rho} \ln \left(1 + \sum_{i=1}^{m} \exp(p g_i(x))\right).
\]

Zhang et al\(^3\) proposed another penalty term function in the form

\[
\frac{1}{\rho} \sum_{i=1}^{m} \exp(p g_i(x)).
\]

Also there are two penalty parameters \(\rho\) and \(p\) in the exact penalty function

\[
f(x) + \rho \frac{1}{\rho} \sum_{i=1}^{m} \exp(p g_i(x)).
\]

Sun and Li\(^4\) presented an exact penalty function in the form

\[
Q_p(x, \lambda) = \frac{1}{\rho} \ln \left[\frac{1}{\rho} \left(\exp(p f(x)) + \sum_{i=1}^{m} \exp(\rho \lambda g_i(x))\right)\right],
\]

where \(\lambda \in \mathbb{R}^m\) and \(p > 0\) are the penalty parameters. In fact the parameters \(\lambda_i\) can take the same value, and thus this penalty function can be regarded as a penalty function with two penalty parameters.

In this paper, we focus on different general penalty functions in integer programming as in\([1]\). We firstly study a general penalty function with one penalty pa-
rameter and then a general penalty function with two penalty parameters. The corresponding exact penalty results are obtained respectively.

The organization of the present paper is as follows. In Section 2, we study the general penalty function with one parameter, and show that any optimal solution of the penalty problem is an optimal solution of \((P)\) when the penalty parameter is sufficiently large. The general penalty function with two parameters is studied in Section 3. We also show that any optimal solution of the penalty problem with two parameters is an optimal solution of \((P)\) when the penalty parameters are sufficiently large.

2 Exact Penalty Function with One Penalty Parameter

Let

\[ \alpha_i = \min_{x \in X_i} g_i(x), \quad i = 1, \ldots, m, \]

\[ \beta = \min_{x \in X} \min_{i \in I^*(x)} g_i(x), \]

\[ \eta = \min_{x \in S^* \cap G(P)} f(x) - f(x^*), \]

where \( I^*(x) = \{ i \mid g_i(x) > 0, i = 1, \ldots, m \}, x^* \in G(P). \) Obviously we have \( \beta > 0, \eta > 0. \)

The general penalty function with one penalty parameter is of the form

\[ T_\mu(f(x), g_1(x), \ldots, g_m(x)), \]

where \( T_\mu : \mathbb{R}^{m+1} \to \mathbb{R}, \) and \( \mu \) is the penalty parameter. Assume that \( T_\mu(y_0, y_1, \ldots, y_m) \) satisfies the following conditions:

1° For any \( \mu > 0, T_\mu(y_0, y_1, \ldots, y_m) \) is increasing with respect to each \( y_i, i = 1, \ldots, m; \)

2° For any \( (y_1, \ldots, y_m) \) with at least one positive element, it holds

\[ T_\mu(y_0, y_1, \ldots, y_m) \to + \infty (\mu \to + \infty). \]

3° For any \( (y_0, y_1, \ldots, y_m) \) with \( y_0 > 0, y_i \leq 0, i = 1, \ldots, m, \) it holds

\[ T_\mu(y_0, y_1, \ldots, y_m) \to y_0 (\mu \to + \infty). \]

Let \( Q(x, \mu) = T_\mu(f(x), g_1(x), \ldots, g_m(x)) \). Then the corresponding penalty problem is of the following form

\[ (P_\mu) \quad \min Q(x, \mu) \]

s.t. \( x \in X_1. \)

Denote the set of global minima of \((P_\mu)\) by \( G(P_\mu)\).

**Lemma 2.1** For any \( i \in \{1, \ldots, m\}, \) there exists a \( \bar{\mu}_i > 0, \) such that

\[ T_\mu(a_0, a_1, \ldots, a_{i-1}, 0, a_{i+1}, \ldots, a_m) > f(x^*) + \frac{\eta}{2} \]

when \( \mu > \bar{\mu}_i. \)

**Proof** From 2°, for any \( i \in \{1, \ldots, m\}, \) we have

\[ T_\mu(a_0, a_1, \ldots, a_{i-1}, 0, a_{i+1}, \ldots, a_m) \to + \infty (\mu \to + \infty). \]

Thus there exists a \( \bar{\mu}_i > 0, \) such that

\[ T_\mu(a_0, a_1, \ldots, a_{i-1}, 0, a_{i+1}, \ldots, a_m) > f(x^*) + \frac{\eta}{2} \]

when \( \mu > \bar{\mu}_i. \)

**Lemma 2.2** Suppose \( x^* \in G(P). \) Then there exists a \( \mu_0 > 0, \) such that

\[ T_\mu(f(x^*), g_1(x^*), \ldots, g_m(x^*)) < f(x^*) + \frac{\eta}{2} \]

when \( \mu > \mu_0. \)

**Proof** By \( x^* \in G(P), \) we have \( g_i(x^*) \leq 0, i = 1, \ldots, m; \) thus by 3°,

\[ T_\mu(f(x^*), g_1(x^*), \ldots, g_m(x^*)) \to f(x^*). \]

Therefore, there exists a \( \mu_0 > 0, \) such that

\[ T_\mu(f(x^*), g_1(x^*), \ldots, g_m(x^*)) < f(x^*) + \frac{\eta}{2} \]

when \( \mu > \mu_0. \)

**Lemma 2.3** Suppose \( x^* \in G(P). \) Then there exists a \( \mu_1 > 0, \) such that

\[ T_\mu(f(x^*) + \eta, a_1, \ldots, a_m) > f(x^*) + \frac{\eta}{2} \]

when \( \mu > \mu_1. \)

**Proof** Since \( S_I \neq \emptyset, \) we have \( a_i \leq 0, i = 1, \ldots, m. \) By 3°, it holds

\[ T_\mu(f(x^*) + \eta, a_1, \ldots, a_m) \to f(x^*) + \eta (\mu \to + \infty). \]

Thus there exists a \( \mu_1 > 0, \) such that

\[ T_\mu(f(x^*) + \eta, a_1, \ldots, a_m) > f(x^*) + \frac{\eta}{2} \]

when \( \mu > \mu_1. \)

Let \( \mu = \max \{ \bar{\mu}_1, \ldots, \bar{\mu}_m, \mu_0, \mu_1 \}. \)

**Theorem 2.1** Suppose that \( T_\mu \) satisfies the conditions 1°, 2° and 3°. Then \( G(P_\mu) \subseteq G(P) \) for \( \mu > \mu. \)

**Proof** We firstly show that \( G(P_\mu) \subseteq S_I. \) By contra-