On Closure Properties of NBU(2) Class of Life Distributions

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Abstract Some new results about the NBU(2) class of life distributions were obtained. Firstly, it was proved that the decrease with time of the increasing concave ordering of the excess lifetime in a renewal process leads to the NBU(2) property of the interarrival times. Secondly, the NBU(2) class of life distributions is proved to be closed under the formation of series systems. Finally, it was also shown that the NBU(2) class is closed under convolution operation.

Key words renewal process, NBU(2), convolution, series system, increasing concave order.

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1 Introduction

Several extensions of the NBU (new better than used) class of life distributions have been proposed in literature. For definitions of such classes of life distributions, like NBUE, HNBUE and their duals, see Shaked and Shanthikumar\(^1\), Barlow and Proschan\(^2\). Let \(X\) be a nonnegative random variable with continuous distribution function \(F(t)\) and survival function \(G(t) = 1 - F(t)\), representing the life length of a unit. For two lives \(X\) and \(Y\), \(X\) is said to be smaller than \(Y\) in the stochastic order (denoted by \(X \leq_{st} Y\)) if \(G(x) < F(x)\), for all \(x > 0\). We denote by \(X_t\), the residual life of a used unit at time \(t > 0\) provided it is working at time \(t\), that is, \(X_t = X - t\) \(X > t\).

It is well known that \(F \in \text{NBU} \iff X_t \leq_{st} X\), for all \(t \geq 0\). As a generalization, Deshpande, Kochar, and Singh\(^3\) suggested the NBU(2) (new better than used of second order) class, and emphasized that \(X_t\) is smaller than \(X\) in increasing concave ordering (denoted as \(X \leq_{ic} X\)), for all \(t > 0\). Cao and Wang\(^4\) proposed the NBUC (new better than used in convex ordering) class, which states that \(X_t\) is smaller than \(X\) in increasing convex ordering (denoted as \(X_t \leq_{ic} X\)), for all \(t > 0\). In economics, the usual stochastic ordering \(\leq_{st}\), the increasing convex ordering \(\leq_{ic}\), and the increasing concave ordering \(\leq_{ic}\) are called the FSD (first order stochastic dominance) ordering, SSD1 (second order stochastic dominance (1)) ordering, and SSD2 (second order dominance (2)) ordering, respectively. They are often used to make a comparison between two random prospects. For definitions and details regarding these orderings, see Chapter 3 of Shaked and Shanthikumar\(^1\). Of course, both extensions include the NBU class as a subclass and are included in the NBUE class.

For the sake of completeness, we present the following notions, and by “decreasing” and “increasing” we refer to nonincreasing and nondecreasing, respectively in the sequel.

Definition 1 Let \(X\) and \(Y\) be two nonnegative random variables with survival functions \(F\) and \(G\), respectively. \(X\) is said to be smaller than \(Y\) in increasing concave ordering, denoted as \(X \leq_{ic} Y\), if for all \(t, y \geq 0\),

\[
\int_0^y F(x)\,dx \leq \int_0^y G(x)\,dx.
\]

Definition 2 Let \(X\) be a life (that is, a nonnegative random variable) with distribution function \(F\). \(F\) is said to be new better than used of second order (denoted as \(X (or F) \in \text{NBU}(2)\)) if for all \(t \geq 0\),

\[
X_t \leq_{ic} X.
\]

Inequality (1) is also equivalent to that, for all \(t, y \geq 0\),

\[
\int_0^y F(x)\,dx \leq \int_0^y G(x)\,dx
\]

or
The dual notion of a new worse than used of second order (NWU(2)) life distribution function is defined by reversing the inequality in \( (1) \). Note that \( F \) is NBU if \( X_t \leq X \) for all \( t \geq 0 \), that is, for all \( x, t \geq 0 \), \( F(t + x) \leq F(t) F(x) \). Directly integrating both sides of this inequality yields \( (1) \), and the NBUE property can be followed by letting \( y \to \infty \) in \( (2) \). Therefore, we have the following chain of implications:

\[
\text{NBU} \Rightarrow \text{NBU(2)} (\text{NBUC}) \Rightarrow \text{NBUE}.
\]

**Definition 3** A stochastic process \( X(t) \) is said to be stochastically decreasing in \( t \geq 0 \) in increasing concave order (denoted as \( X(t) \downarrow \infty \) in \( t \geq 0 \)), if, for all \( 0 \leq s \leq t \) and \( y \geq 0 \),

\[
\int_0^y P(X(t) > x) \, dx \leq \int_0^y P(X(s) > x) \, dx.
\]

(4)

Obviously, \( (4) \) is equivalent to, for all \( 0 \leq s \leq t \), \( X(t) \leq \omega X(s) \).

A stochastic process \( X(t) \) is said to be stochastically increasing in increasing concave ordering when the inequality \( (4) \) is reversed. Correspondingly, \( X(t) \) is said to be stochastically decreasing (increasing) in increasing convex ordering when we substitute \( [0, y] \) for the interval \( [y, \infty) \) in the inequality \( (4) \).

**2 A Sufficient Condition Characterized by Residual Life Function**

Let \( \{X_n, n = 1, 2, \cdots\} \) be a sequence of mutually independent and identically distributed nonnegative random variables with common life distribution \( F \). For the renewal process with \( X_i's \) as its interarrival times, denote, for \( n = 1, 2, \cdots \), \( S_n = \sum_{i=1}^n X_i \), the time of the \( n \)-th arrival, and \( S_0 = 0 \). Again, let \( N(t) = \sup \{n : S_n \leq t \} \) represent the number of events that occur before or at time \( t \). Then \( \{ N(t), t \geq 0 \} \) is called a renewal counting process. Define \( \gamma(t) \) to be residual life at time \( t \), that is,

\[
\gamma(t) = S_{N(t)} - t.
\]

Chen\(^{[5]}\) proved that if \( \gamma(t) \) is stochastically decreasing in, then \( F \) is NBU, it was also shown in that paper that, if \( E[\gamma(t)] \) is decreasing in \( t \) and \( E[\gamma(0)] = E[X_t] < \infty \), then \( F \) is NBUE. Li et al.\(^{[6]}\) proved that, if \( \gamma(t) \) is decreasing in \( t \) in increasing convex ordering, then \( F \) is NBUC. In this section, an analogous result about the NBU(2) life distributions is obtained.

**Theorem 1** If \( \gamma(t) \) is decreasing (increasing) in \( t \) in increasing concave ordering, then \( F \) is NBU(2) (NWU(2)).

**Proof** Let \( g(t, x) = P(\gamma(t) > x) = \bar{F}_{\gamma(t)}(x) \), for \( t \geq 0 \) and \( x \geq 0 \). By total probability, it follows that

\[
g(t, x) = \bar{F}(t + x) + \int_0^t g(t - s, x) \, dF(s).
\]

(5)

For more details see Karlin and Taylor\(^{[7]}\). Integrating both sides of \( (5) \) gives, for all \( y \geq 0 \),

\[
\int_0^y g(t, x) \, dx = \int_0^y \bar{F}(t + x) \, dx + \int_0^y \int_0^t g(t - s, x) \, dF(s) \, dx = I_1 + I_2.
\]

Since \( \gamma(t) \) is decreasing in \( t \) according to increasing concave ordering, we get

\[
I_2 = \int_0^y \int_0^t g(t - s, x) \, dx \, dF(s)
\]

\[\geq \int_0^t \int_0^y g(t, x) \, dx \, dF(s)\]

\[= \int_0^y \int_0^t g(t - s, x) \, dF(s) \, dx = F(t) \int_0^y g(t, x) \, dx.
\]

Thus, we have, for all \( y \geq 0 \),

\[
\int_0^y \bar{F}(t + x) \, dx \leq \bar{F}(t) \int_0^y g(t, x) \, dx.
\]

(6)

Since \( \gamma(t) \downarrow \infty \) in \( t \geq 0 \), we have for all \( t \geq 0 \), \( \gamma(t) \leq \omega \gamma(0) = X_t \), that is, for all \( y \geq 0 \) and \( t \geq 0 \),

\[
\int_0^y g(t, x) \, dx \leq \int_0^y \bar{F}(x) \, dx.
\]

(7)

From \( (6) \) and \( (7) \), it follows that, for all \( t \geq 0, y \geq 0 \),

\[
\int_0^y \bar{F}(t + x) \, dx \leq \bar{F}(t) \int_0^y \bar{F}(x) \, dx.
\]

That is, \( F \) is NBU(2).

When \( \gamma(t) \) is increasing in \( t \) in increasing concave ordering, we can prove that \( F \) is NWU(2) by reversing the inequalities above.

Combining Theorem 1 and Theorem 2 of Chen\(^{[5]}\) and Theorem 7 of Li, et al.\(^{[6]}\) with the present Theorem 1, we get the following implications: