Three-Dimensional Vibration of Thick Plate

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Abstract By the method of initial functions (MIF) and based upon the basic equations of three-dimensional theory of elasto-dynamics, the governing differential equations of plate with arbitrary thickness are formulated in this paper. The dynamic response of stress and displacement of thick plate subjected to the transverse forces is obtained. It is shown that the vibration characteristics of thick plate consist of three modes: thickness shear mode, symmetric mode and anti-symmetric mode. The characteristic equations of simply supported thick plate are derived and the comparison of the free vibration frequencies of moderate thick plate theory and three-dimensional elasticity theory with the classical theory is made.

Keywords thick plate, free vibration, forced vibration

1 Introduction

The classical thin plate theory of vibration is based on the Kirchhoff hypothesis, and the effect of transversal shear deformation and rotatory inertia are neglected. The free vibration frequencies of thin plates with all kinds of boundary conditions have been given in [1]. For the analysis of the vibration problems of thick plate, the moderate-thick plate theory of Reissner, Mindlin and Henky have been used for years. For various effects of transversal shear stress, transversal normal stress and rotatory inertia, some dynamic problems of thick plate in engineering have been solved in [2].

Both the classical theory and the moderate thick plate theory are approximate and based on the hypothesis. In recent years, many domestic or foreign researchers [3–4] have achieved a series of results on the static, dynamic and stability problems of thick plate, based on three-dimensional elasticity theory.

The MIF was first put forward by Vlasov [5]. In this paper, the state equations will be derived from the three-dimensional elasticity theory by means of MIF. By decomposing the transverse forces, the vibration characteristics of thick plate are obtained. This paper shows that the vibration characteristics of the problem consist of thickness shear mode, symmetric and anti-symmetric modes. Those results cannot be got from the classic plate theory. As an example, the characteristic equations of free vibration of simply supported thick plate are deduced, and the numerical results are given.

2 Formulation of State Equations

If the displacement components and transversal stress components are regarded as the fundamental unknown quantities, the governing equation of the isotropic thick plate can be written as follows:

\[
Y\{T_1\} = [A]\{T_2\},
\]

\[
Y\{T_1\} = [B]\{T_2\},
\]

where \( Y = \partial / \partial z \), \( \{T_1\} = [U \ V \ Z]^T \), \( \{T_2\} = [X \ Y \ W]^T \),

\[
[A] = \begin{bmatrix}
1 & 0 & -\sigma \\
0 & 1 & -\beta \\
-\alpha & -\beta & 1
\end{bmatrix},
\]

\[
[B] = \begin{bmatrix}
\xi_x - \beta^2 - \frac{2}{1-\mu} \xi_z & -\frac{1+\mu}{1-\mu} \xi_z^2 & -\frac{\mu}{1-\mu} \\
-\frac{1+\mu}{1-\mu} \xi_z & -\xi_z^2 - \frac{2}{1-\mu} \beta^2 & -\frac{\mu \beta}{1-\mu} \\
-\frac{\mu}{1-\mu} & -\frac{\mu \beta}{1-\mu} & 1 - \frac{2 \mu}{1-\mu}
\end{bmatrix},
\]

and \( U = Gu \), \( V = Gv \), \( W = Gw \), \( X = r_x \), \( Y = r_y \), \( Z = \sigma \),

\[
\partial / \partial x = \alpha, \quad \partial / \partial y = \beta, \quad \rho \frac{\partial^2}{C \partial t^2} = \frac{1}{C} \frac{\partial^2}{\partial t^2} = \xi_z,
\]

in which \( C \) is the velocity of shear wave, and \( G, \mu, \rho \) are shear modulus, Poisson's ratio and the mass density, respectively. The remaining stresses are taken as
3 The Solution of State Equation

From Eq. (1), the following expressions can be obtained:

\[
\begin{align*}
\gamma^2(T_1) &= [A][B](T_2) = [E](T_1), \\
\gamma^2(T_2) &= [B][A](T_1) = [F](T_2),
\end{align*}
\]

and the solutions of state equations (1) can be written as the McLaurin series in the z-direction

\[
\{T\} = \sum_{i=0}^{\infty} \left( \frac{z}{i!} \right)^i \{T_i\},
\]

where \(\{T_i\} = \{T_1, T_2\}\). By using the Sylvester theorem and the series of hyperbolic functions, Eq. (4) can be expressed as

\[
\begin{align*}
\frac{\phi_1}{\gamma_1} + \frac{\phi_2}{\gamma_2} &+ \frac{\phi_3}{\gamma_3} + \cdots = \sum_{i=0}^{\infty} \left( \frac{z}{i!} \right)^i \{T_i\},
\end{align*}
\]

where \(\gamma_i = \lambda_i^2 - \gamma^2\), \(\gamma_i = \lambda_i^2 \gamma^2\), \(\lambda_1 = \lambda_2 = -2\mu(1-\nu)\), \(\lambda_2 = \lambda_2 = \Omega(1-\nu)\)

4 Governing Equations of Vibration Problems of Thick Plate

The loads \(p_1(x,y,t)\) and \(p_2(x,y,t)\) acting on the lateral surfaces are decomposed into symmetric load \(p_a(x,y,t)\) and antisymmetric load \(p_a(x,y,t)\) (Fig. 1), where

\[
p_a = (p_1 + p_2)/2, \quad p_a = (p_1 - p_2)/2.
\]

4.1 Governing equation of vibration of plate subjected to the symmetric loads

The initial functions, in the middle plane \(z=0\), are

\[
X_0 = Y_0 = W_0 = 0,
\]

and the boundary conditions at the lateral surfaces are

\[
z = \pm h, \quad X = Y = 0, \quad Z = -p_a(x,y,t).
\]

Substituting (7) and (8) into the state equations (5) yields

\[
[L_{ij}]_{n=3} (U_0 V_0 Z_0) = (-p_a 0 0)^T, \quad (i=3,4,5, j=1,2,3).
\]

By introducing an auxiliary function \(\phi(x,y,t)\), the governing equation can be expressed as

\[
\begin{align*}
\frac{\phi}{\gamma_1} + \frac{\phi}{\gamma_2} &+ \frac{\phi}{\gamma_3} + \cdots = \sum_{i=0}^{\infty} \left( \frac{z}{i!} \right)^i \{T_i\},
\end{align*}
\]

and the solutions of state equations (1) can be written as the McLaurin series in the z-direction

\[
\{T\} = (1+z\gamma^2+y_2^2+y_3^2+\cdots) \{T\} |_{z=0},
\]

where \(\{T\} = [T_1, T_2]^T\). By using the Sylvester theorem and the series of hyperbolic functions, Eq. (4) can be expressed as

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where \(\gamma_i = \lambda_i^2 - \gamma^2\), \(\gamma_i = \lambda_i^2 \gamma^2\), \(\lambda_1 = \lambda_2 = -2\mu(1-\nu)\), \(\lambda_2 = \lambda_2 = \Omega(1-\nu)\)

4.2 Governing equation of vibration of plate subjected to the antisymmetric loads

The initial functions in the middle plane \(z=0\) are

\[
X_0 = Y_0 = W_0 = 0,
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and the boundary conditions at the lateral surfaces are

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