TRAVELING WAVE SOLUTIONS TO BEAM EQUATION WITH FAST-INCREASING NONLINEAR RESTORING FORCES

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Abstract. On studying traveling waves on a nonlinearly suspended bridge, the following partial differential equation has been considered:

\[ u_{ttt} + u_{xxxx} + f(u) = 0, \]

where \( f(u) = u^+ - 1 \). Here the bridge is seen as a vibrating beam supported by cables, which are treated as a spring with a one-sided restoring force. The existence of a traveling wave solution to the above piece-wise linear equation has been proved by solving the equation explicitly (McKenna & Walter in 1990). Recently the result has been extended to a group of equations with more general nonlinearities such as \( f(u) = u^+ - 1 + g(u) \) (Chen & McKenna, 1997). However, the restrictions on \( g(u) \) do not allow the resulting restoring force function to increase faster than the linear function \( u - 1 \) for \( u > 1 \). Since an interesting “multiton” behavior, that is, two traveling waves appear to emerge intact after interacting nonlinearly with each other, has been observed in numerical experiments for a fast-increasing nonlinearity \( f(u) = e^{u^+ - 1} \), it hints that the conclusion of the existence of a traveling wave solution with fast-increasing nonlinearities shall be valid as well.

In this paper, the restoring force function of the form \( f(u) = u^+ h(u) - 1 \) is considered. It is shown that a traveling wave solution exists when \( h(u) \geq 1 \) for \( u \geq 1 \) (with other assumptions which will be detailed in the paper), and hence allows \( f \) to grow faster than \( u - 1 \). It is shown that a solution can be obtained as a saddle point in a variational formulation. It is also easy to construct such fast-increasing \( f(u) \)'s for more numerical tests.

§ 1 Introduction

It has been found by McKenna and Walter\(^{1}\) that traveling wave solutions to the following nonlinear beam equation exist

\[ u_{ttt} + u_{xxxx} + u^+ - 1 = 0. \]
Equation (1) has been posed as a simple model of a nonlinearly suspended bridge, where $u^+ = \max\{u, 0\}$ represents a one-sided restoring force and $f(u) = u^+ - 1$ is hence called the restoring force function.

The strategy of finding the traveling wave solutions is to study the partial differential equation on the real line, look for solutions of the form $u(x, t) = v(x - ct) + 1$, and hence turn to solve the following ordinary differential equation

$$v^{(4)} + c^2v'' + (v + 1)^+ - 1 = 0,$$

where $v(x)$ decays to zero exponentially at infinities.

The existence of traveling wave solutions to (1) has been proved by McKenna and Walter first, by a purely calculus approach \cite{1}. That is, the two linear ordinary differential equations have been solved explicitly for $v > -1$ and $v < -1$, and then the two pieces of solutions have been matched at $v = -1$. However, since this proof depends on the explicit solutions of the piece-wise linear ordinary differential equation, it may become invalid with the slightest perturbation in the nonlinearity.

In \cite{2}, this problem has been solved in the sense that a qualitative proof based on the Mountain Pass Lemma \cite{3, 4} has been given. Furthermore, it has been shown that the ordinary differential equation

$$v^{(4)} + c^2v'' + f(v + 1) = 0$$

has at least one non-trivial solution provided that $f(v+1)$ behaves "like" $(v+1)^+ - 1$ in a qualitative way. However, it shall be noticed that the restrictions on $f(u)$ given in \cite{2} only allow this function to increase slower than $u^+ - 1$ for $u > 1$. This is a serious limitation which leaves out of account many fast-increasing analytic functions.

Actually, in the further investigation of Chen and McKenna \cite{2, 5}, an interesting phenomenon has been observed when the nonlinearity blows up much faster than $u^+ - 1$, namely $f(u) = e^{u-1} - 1$. According to the results of the numerical experiments, some of the traveling wave solutions appear to be so stable that when two of these waves collide, they pass through each other like solitons. These waves were hence named "multitons", a term suggested by Louis Nirenberg to describe these many-noded traveling waves.

In this paper, the equation

$$u_{tt} + u_{xxxx} + f(u) = 0$$

is discussed where the original piece-wise linear restoring force function is replaced by $f(u) = u \cdot h(u) - 1$. Here $h(u)$ is an analytic function which shares some of the qualitative behavior of the Heaviside function

$$H(u) = \begin{cases} 1 & \text{if } u \geq 0, \\ 0 & \text{if } u < 0. \end{cases}$$

It will be shown that the conclusion on the existence of a non-trivial traveling wave solution is also valid for $f(u)$ blowing up faster than $u^+ - 1$.

The rest of this paper will consist of three sections. The first will briefly describe the