A CLASS OF INFINITE MATRIX TOPOLOGICAL ALGEBRAS

Wu Junde

Abstract. Let $\lambda$ and $\mu$ be sequence spaces and have both the signed-weak gliding hump property, $(\lambda, \mu)$ the algebra of the infinite matrix operators which transform $\lambda$ into $\mu$. In this paper, it is proved that if $\lambda$ and $\mu$ are $\beta$-spaces and $\lambda'$ and $\mu'$ have also the signed-weak gliding hump property, then for any polar topology $r$, $((\lambda, \mu), r)$ is always sequentially complete locally convex topological algebra.

§ 1 Introduction

Let $\omega$ be the space of all scalar valued sequences, $c$ the space of convergent sequences and $\hat{c}$ the space with only finitely many non-zero coordinates.

We say a non-zero vectors sequence $\{z^{(n)}\}$ in $\omega$ is a block sequence if there exists a strictly increasing sequence of positive integers $\{k_n\}$ such that

$$z^{(n)} = (0, \ldots, 0, z_{k_n-1+1}^{(n)}, \ldots, z_k^{(n)}, 0, \ldots).$$

$E$ is said to have the signed-weak gliding hump property (S-WGHP) if given any $x \in E$ and any block sequence $\{x^{(k)}\}$ with $x = \sum_{k=1}^{\infty} x^{(k)}$ (pointwise sum), then for each strictly increasing positive integers sequence $\{m_k\}$ has a further subsequence $\{n_k\}$ and a sequence $\{h_k\}$ with $h_k = 1$ or $-1$ $(k\in\mathbb{N})$ such that $\tilde{x} = \sum_{k=1}^{\infty} h_k x^{(n_k)} \in E$ (pointwise sum) (see [1]).

The $\beta$-dual of a sequence space $E$ is defined by

$$E^\beta = \{x \in \omega; \sum_{i=1}^{\infty} x_i y_i \text{ is convergent for each } y \in E\}.$$ 

If $E^\beta = E$, $E$ is called a $\beta$-space (see [2, 3]). If $E \supseteq \phi$, then $E$ and $E^\beta$ are in duality with respect to the bilinear pairing $\langle x, y \rangle = \sum_{i=1}^{\infty} x_i y_i$, for $x \in E, y \in E^\beta$. We denote the weak topology on $E$ from this pairing by $\sigma(E, E^\beta)$, similar notation is used for the weak topology on $E^\beta$.

Received: 2000-04-21.


Keywords: Sequence space, infinite matrix, topological algebra.

This research is partly supported by the NSF of Hei Longjiang.
§ 2 The Topological Algebra $((\lambda, \mu), (A, \rho))$

Let $\lambda, \mu$ be two sequence spaces and $\lambda \supset \phi$ has the S-WGHP, $A = (a_{ij})$ a scalar valued infinite matrix. If for each $x = (x_j) \in \lambda$ and $i \in \mathbb{N}$, the series $\sum_{j=1}^{\infty} a_{ij} x_j$ is convergent and $\\left\{ \sum_{j=1}^{\infty} a_{ij} x_j \right\}_{i=1}^{\infty} \in \mu$, then we say that $A$ transforms $\lambda$ into $\mu$. Let $(\lambda, \mu)$ denote the all scalar valued matrices which transform $\lambda$ into $\mu$.

Recently, Wu Junde et al. showed that if $A \in (\lambda, \mu)$, then for any $x \in \lambda$ and $t \in \mu^\lambda$,

$$\sum_{i=1}^{\infty} t_i \sum_{j=1}^{\infty} a_{ij} x_j = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} t_j x_j = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij} x_j t_i.$$  

That is,

$$t' (Ax) = x' (A't), x \in \lambda, t \in \mu^\lambda. \quad (#)$$

Where $x$, $t$ are column vectors and $x'$, $t'$ are row vectors, $A'$ is the transposed matrix of $A$. Furthermore, if $\lambda \supset \mu \supset \phi$ and $\mu$ has also the S-WGHP, then $(\lambda, \mu)$ is an algebra.

Now, we equip $(\lambda, \mu)$ with the polar topology $\tau$ as follows:

Let $\mathcal{U}$ be a family of $\sigma(\lambda, \beta)$ bounded sets and $\mathcal{D}$ a family of $(\mu^\beta, \mu)$ bounded sets such that $\bigcup_{M \in \mathcal{U}} M = \lambda$, $\bigcup_{D \in \mathcal{D}} D = \mu^\beta$. The polar topology $\tau$ is determined by the neighbourhoods basis $V_r(\theta, M, D)$ of $\theta$ in $(\lambda, \mu)$:

$$V_r(\theta, M, D) = \{ A \in (\lambda, \mu), \sup_{u \in D, x \in M} |u(Ax)| \leq 1 \}.$$  

If $\mathcal{D}$ and $\mathcal{U}$ are the all finite sets of $\mu^\beta$ and $\lambda$, respectively, then $\tau$ is said to be the weak topology, and denoted by $\tau_\beta$. It is easily to prove that for any polar topology $\tau$ of $(\lambda, \mu)$, $((\lambda, \mu), \tau)$ is a locally convex topological algebra ($[4]$).

§ 3 The Sequentially Completeness of $((\lambda, \mu), (A, \rho))$

Lemma 1. If $\lambda \supset \phi$ and $\lambda$ has the S-WGHP, $A = (a_{ij}) \in (\lambda, \rho)$, then for each $(x_j) \in \lambda$, the series $\sum_{j=1}^{\infty} a_{ij} x_j$ converges uniformly with respect to $i \in \mathbb{N}$. Furthermore, if $\lim_{i} a_{ij} = a_j$, then the series $\sum_{j=1}^{\infty} a_j x_j$ also converges and $\lim_{i} \sum_{j=1}^{\infty} a_{ij} x_j = \sum_{j=1}^{\infty} a_j x_j$.

Now, we at first establish the sequentially completeness theorem of $((\lambda, \mu), \tau_\beta)$ as follows:

Theorem 1. Let $\lambda \supset \mu \supset \phi$ and $\lambda$ and $\mu$ have both the S-WGHP, $\lambda$ and $\mu$ be $\beta$-spaces. If $\lambda^\beta$ and $\mu^\beta$ have also the S-WGHP, then $((\lambda, \mu), \tau_\beta)$ is sequentially complete.

Proof. Let $\{ A^{(n)} = (a_{ij}^{(n)}) \} \in (\lambda, \mu)$ be a $\tau_\beta$-Cauchy sequence, since $\lambda \supset \phi, \mu^\beta \supset \phi$, so for each $i, j \in \mathbb{N}$, $(a_{ij}^{(n)})$ is a convergent sequence. Denote $\lim_{n} a_{ij}^{(n)} = a_{ij}, A = (a_{ij})$, and