ISOMETRIC ISOMORPHISMS OF TRIANGULAR BANACH ALGEBRAS

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**Abstract.** Let \( \mathcal{A} \) and \( \mathcal{B} \) be Banach algebras. Let \( \mathcal{M} \) be a Banach \( \mathcal{A}, \mathcal{B} \)-module with bounded 1. Then \( \mathcal{F} = \begin{bmatrix} \mathcal{A} & \mathcal{M} \\ 0 & \mathcal{B} \end{bmatrix} \) is a Banach algebra with the usual operations and the norm

\[
\left\| \begin{bmatrix} A & M \\ 0 & B \end{bmatrix} \right\| = \| A \| + \| M \| + \| B \|.
\]

Such an algebra is called a triangular Banach algebra. In this paper the isometric isomorphisms of triangular Banach algebras are characterized.

**§ 1 Introduction**

Let \( \mathcal{A} \) and \( \mathcal{B} \) be Banach algebras. Let \( \mathcal{M} \) be a Banach \( \mathcal{A}, \mathcal{B} \)-module with bounded 1. That is, \( \| AMB \| \leq \| A \| \| M \| \| B \| \) for each \( A \in \mathcal{A}, B \in \mathcal{B} \) and \( M \in \mathcal{M} \). Then

\[
\mathcal{F} = \begin{bmatrix} \mathcal{A} & \mathcal{M} \\ 0 & \mathcal{B} \end{bmatrix} = \left\{ \begin{bmatrix} A & M \\ 0 & B \end{bmatrix} : A \in \mathcal{A}, B \in \mathcal{B}, M \in \mathcal{M} \right\}
\]

is a Banach algebra with the usual operations and the norm

\[
\left\| \begin{bmatrix} A & M \\ 0 & B \end{bmatrix} \right\| = \| A \| + \| M \| + \| B \|.
\]

We call such an algebra a **triangular Banach algebra**. In [1], the derivation of triangular Banach algebras is studied and the first cohomology groups of some special triangular Banach algebras are explicitly determined. We have seen the triangular Banach algebras provide a rich array of examples of Banach algebras with nontrivial cohomology.

In this paper, we study the isometric isomorphisms of triangular Banach algebras. Isometries of Banach algebras were studied by many mathematicians. Kadison\(^{[3]}\) obtained a famous characterization of a surjective, linear, isometric maps from one \( C^* \)-algebra onto another. In \([3, 4]\), authors of these papers independently proved that a linear surjective isometric map from a nest algebra onto another is of form \( A \to U AW \) or \( A \to UJA^* JW \), where \( U \) and \( W \) are suitable unitary operators and \( J \) is a fixed involution of a Hilbert space. In \([5]\), it is proved that a linear surjective isometric map acting on sub-strongly reducible
maximal triangular algebras is also of the stated form. By these results, it is easily seen that the isometric isomorphism of the algebras concerned above is spatially implemented by a unitary operator. In this paper, we characterize isometric isomorphisms of triangular Banach algebras.

§ 2 Isometric Isomorphisms of Triangular Banach Algebras

In this section, we pay our attention to triangular Banach algebras \( T = \begin{bmatrix} \mathcal{A} & \mathcal{M} \\ 0 & \mathcal{B} \end{bmatrix} \) where \( \mathcal{A} \) and \( \mathcal{B} \) are the unitary Banach algebras and \( \mathcal{M} \) is \( \mathcal{A}, \mathcal{B} \)-module. We always assume that \( \phi \) is an isometric isomorphism from \( T \) onto itself.

**Lemma 2.1.** If \( J \) is a projection in \( \mathcal{A} \) (i.e. \( J^2 = J \) and \( \| J \| = 1 \)), then \( \phi \left( \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} \right) \) equals \( \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix} \) or \( \begin{bmatrix} 0 & 0 \\ 0 & Q \end{bmatrix} \), where \( P \) and \( Q \) are projections in \( \mathcal{A} \) and \( \mathcal{B} \) respectively.

**Proof.** Suppose that
\[
\phi \left( \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} P & M \\ 0 & Q \end{bmatrix},
\]
then
\[
\| P \| + \| Q \| + \| M \| = 1. \tag{2.1}
\]
Since
\[
\begin{bmatrix} P & M \\ 0 & Q \end{bmatrix} = \phi \left( \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} \right) = \phi \left( \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} \right) \phi \left( \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} P^2 & PM + MQ \\ 0 & Q^2 \end{bmatrix},
\]
both \( P \) and \( Q \) are idempotent. Thus

1. If \( P \neq 0 \), then \( \| P \| \geq 1 \). By Equality (2.1) we have \( \| P \| = 1 \) and \( Q = M = 0 \);
2. If \( Q \neq 0 \), then \( \| Q \| \geq 1 \). By Equality (2.1) we have \( \| Q \| = 1 \) and \( P = M = 0 \);
3. If \( P = Q = 0 \), then \( M = PM + MQ = 0 \). This conflicts with the injectivity of \( \phi \).

**Lemma 2.2.** \( \phi \left( \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \right) \) equals \( \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \) or \( \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \), where \( I \) is a unitary element.

**Proof.** By the lemma above, first suppose that \( \phi \left( \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix} \), where \( P \) is a projection in \( \mathcal{A} \). Suppose that \( \phi \left( \begin{bmatrix} A & M \\ 0 & B \end{bmatrix} \right) = \begin{bmatrix} I - P & 0 \\ 0 & 0 \end{bmatrix} \). Then
\[
\phi \left( \begin{bmatrix} A & M \\ 0 & B \end{bmatrix} \right) = \phi \left( \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A & M \\ 0 & B \end{bmatrix} \right) = \phi \left( \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \right) \phi \left( \begin{bmatrix} A & M \\ 0 & B \end{bmatrix} \right) = \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I - P & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \).