EXISTENCE OF PERIODIC SOLUTIONS OF PLANAR SYSTEMS WITH FOUR DELAYS

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Abstract. The sufficient condition for the existence of non-constant periodic solutions of the following planar system with four delays are obtained:

\[
\begin{align*}
    x_1'(t) &= -a_1 x_1(t) + a_1 f_1(x_1(t - r_1), x_2(t - r_2)), \\
    x_2'(t) &= -b_1 x_2(t) + b_1 f_2(x_1(t - r_3), x_2(t - r_4)).
\end{align*}
\]

This approach is based on the continuation theorem of the coincidence degree, and the a-priori estimate of periodic solutions.

\section{Introduction}

For the planar delay differential system with a single delay in both equations

\[
\begin{align*}
    x_1'(t) &= -x_1(t) + a^* f_1(x_1(t - 1), x_2(t - 1)), \\
    x_2'(t) &= -x_2(t) + a^* f_2(x_1(t - 1), x_2(t - 1)),
\end{align*}
\]  

(1.1)

the first piece of research on non-constant periodic solutions was done in \[1\], where \(a^* > 0\) is a constant, \(f_1\) and \(f_2\) are bounded \(C^3\) functions on \(\mathbb{R}^2\) satisfying

\[
\frac{\partial f_1}{\partial x_2}(0, 0) \neq 0, \quad \frac{\partial f_2}{\partial x_1}(0, 0) \neq 0,
\]

and the negative feedback conditions

\[
\begin{align*}
    x_2 f_1(x_1, x_2) > 0, x_2 \neq 0, \\
    x_1 f_2(x_1, x_2) < 0, x_1 \neq 0.
\end{align*}
\]

(1.2)

In \[1\] it is showed that there exists an \(a_0 > 0\) such that for \(a^* > a_0\), there is a non-constant periodic solution with period greater than 4. Further study on the global existence of periodic solutions to system (1.1) can be found in \[2,3\]. All together there are very few results on the global existence of periodic solutions of planar systems with a delay appearing in both equations, especially the results involving planar systems with two delays are very scarce, we only find in \[4\] the investigation of planar systems with two delays.
\[
\begin{align*}
&x_1'(t) = -a_0x_1(t) + a_1f_1(x_1(t - \tau_1), x_2(t - \tau_2)), \\
&x_2'(t) = -b_0x_2(t) + b_1f_2(x_1(t - \tau_1), x_2(t - \tau_2)),
\end{align*}
\]  \tag{1.3}

where \(a_0 > 0, b_0 > 0, a_1, b_1, a_2, b_2, r_1, r_2, i = 1, 2, 3, 4\) are constants, and \(f_1\) and \(f_2\) satisfy the following assumption:

\[
\begin{align*}
&f_j \in C^1(\mathbb{R}^2), f_j(0, 0) = 0, \quad \frac{\partial f_j}{\partial x_j}(0, 0) = 0, \quad j = 1, 2, \\
&\frac{\partial f_j}{\partial x_i}(0, 0) \neq 0, \quad \frac{\partial f_j}{\partial x_i}(0, 0) \neq 0, \quad x_jf_i(x_1, x_2) \neq 0, \text{ for } x_2 \neq 0, \\
&\text{and } x_jf_i(x_1, x_2) \neq 0, \text{ for } x_1 \neq 0.
\end{align*}
\]  \tag{H_i}

The method of showing the existence of non-constant periodic solution used by the above mentioned researchers came from a widely known idea of Jones\(^{[5]}\). In this seminal paper, Jones introduced the idea of finding a cone in the phase space that maps into itself under a certain operator defined by the flow. The fixed points of this operator are corresponding to periodic solutions of differential equations. The cone is easy to construct, but some other complications arise because most problems have zero as an equilibrium point, which corresponds to the trivial periodic solution. Thus, one needs to find non-zero fixed points of the flow operator knowing in advance that zero is a fixed point. This can be achieved by applying a theorem due to Nussbaum\(^{[6]}\), which depends on the ejectivity of fixed points of the flow operator. For a more applicable form of Nussbaum’s theorem, we refer to \([7,8]\).

Instead of applying Nussbaum’s theorem, in \([4]\) a different approach, the degree theory, was used to study the global existence of periodic solutions of system (1.3). Degree theory has been employed to develop global Hopf bifurcation theory for delay differential equations since the work of \([9]\). By using the global Hopf bifurcation theorem in \([10]\), which was established using a purely topological argument, Ruan and Wei\(^{[4]}\) obtained the result for the existence of non-constant periodic solution of system (1.3).

In the present paper, we consider the following planar system with four delays

\[
\begin{align*}
&x_1'(t) = -a_0x_1(t) + a_i1f_i(x_1(t - \tau_i), x_2(t - \tau_2)), \\
&x_2'(t) = -b_0x_2(t) + b_i1f_i(x_1(t - \tau_i), x_2(t - \tau_i)),
\end{align*}
\]  \tag{1.4}

where \(a_0, b_0, a_1, b_1, \tau_i, i = 1, 2, 3, 4\) are constants, \(a\) is the quotient of two positive odd integers, and \(f_i : \mathbb{R}^2 \to \mathbb{R}, i = 1, 2,\) are continuous functions.

In the paper, instead of applying Nussbaum’s theorem and global Hopf bifurcation theory developed by employing degree theory, we combine the continuation theorem of Mawhin’s coincidence degree\(^{[11]}\) with the differential inequality technique for a priori bounds of non-constant \(w\)-periodic solution \((w > 0)\) of a parametrized planar systems and establish some new sufficient conditions for the existence of non-constant \(w\)-periodic solutions of system (1.4).

Compared with the result obtained in \([4]\) for non-constant periodic solutions of system (1.3), our results in the case when \(a_0 > 0, b_0 > 0, a_1 = 1, \tau_3 = \tau_1\) and \(\tau_4 = \tau_2\) are concise and