SOME RESULTS ON LAG INCREMENTS OF PRINCIPAL VALUE OF BROWNIAN LOCAL TIME

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Abstract. Let $W$ be a standard Brownian motion, and define $Y(t) = \int_0^t ds \int_0^x W(s) ds$ as Cauchy's principal value related to the local time of $W$. We study some limit results on lag increments of $Y(t)$ and obtain various results all of which are related to earlier work by Hanson and Russo in 1983.

§ 1 Introduction

Let $(W(t); t \geq 0)$ be a one-dimensional Brownian motion with $W(0) = 0$, and let $(L(t, x); t \geq 0, x \in \mathbb{R})$ denote its local time process. That is, for any Borel function $f \geq 0$,\[ \int_0^t f(W(s)) ds = \int_{-\infty}^\infty f(x) L(t, x) dx, \quad t \geq 0. \]

We are interested in the following process $(Y(t); t \geq 0)$ which is called Cauchy's principal value related to Brownian local time:

\[ Y(t) = \int_0^t \frac{ds}{W(s)} = \int_{-\infty}^\infty \frac{L(t, x) - L(t, -x)}{x} dx. \] (1.1)

The study of Cauchy's principal value of Brownian local time can be at least traced back to Itô and McKean\cite{9}, and has become very active since the late 1970s, due to applications in various branches of stochastic analysis. An important fact is that the principal values of local times can be represented as the Hilbert transform, or more generally, fractional derivatives, of local times. Also, the principal values of Brownian local times are the key ingredient in establishing Bertoin's excursion theory for Bessel processes of small dimensions\cite{10}. For a detailed account of various motivations, historical facts and general properties of principal values of local times, we refer to the recent collection of research papers in \cite{11} and the survey paper by Yamada\cite{10}. The process $Y(\cdot)$ defined in (1.1) is continuous, having zero quadratic variation, and $Y(\cdot)$ inherits a scaling property from Brownian motion, that is for any fixed $a > 0$.\[ \quad \]

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\[ a^{-\frac{1}{2}}Y(at) \xrightarrow{\text{law}} Y(t). \quad (1.2) \]

In fact, \( Y(\cdot) \) behaves somewhat like a Brownian motion. Hu and Shi\(^7\) studied the global and local almost sure asymptotics of \( Y(\cdot) \), they obtained

\[ \limsup_{t \to \infty} \frac{Y(t)}{\sqrt{t \log \log t}} = \sqrt{8}, \quad \text{a.s.} \quad (1.3) \]
\[ \limsup_{t \to 0} \frac{Y(t)}{\sqrt{t \log \log (1/t)}} = \sqrt{8}, \quad \text{a.s.} \quad (1.4) \]

Csáki et al.\(^5\) determined the modulus of continuity and the large increments of \( Y(\cdot) \),

\[ \limsup_{h \to 0} \sup_{0 \leq s \leq h, 0 \leq t \leq h} \frac{|Y(t+s) - Y(t)|}{\sqrt{\log(1/h)}} = 2, \quad \text{a.s.} \quad (1.5) \]
\[ \limsup_{T \to \infty} \sup_{0 \leq t \leq T, 0 < h \leq T} \frac{|Y(t+s) - Y(t)|}{\sqrt{a_T \log(T/a_T)}} = 2, \quad \text{a.s.} \quad (1.6) \]

where \( a_T \) is the length of time window, which is supposed to satisfy the following condition:

\[ 0 < a_T \leq T, \quad (1.7) \]
\[ a_T \text{ and } T/a_T \text{ are both non-decreasing}, \quad (1.8) \]
\[ \lim_{T \to \infty} \frac{\log(T/a_T)}{\log \log T} = \infty. \quad (1.9) \]

Comparing (1.3)–(1.6) with the corresponding laws for Brownian motion, we see that \( \frac{1}{2}Y(t) \) and \( W(t) \) satisfy exactly the same global and local LIL’s. \( \frac{1}{\sqrt{2}}Y(t) \) and \( W(t) \) have the same moduli of continuity and same increment sizes\(^6\). By (1.1), we know \( Y(t) \) is closely connected with Brownian local time \( L(t,x) \). \([4]\) studied the limit result of the large increments of \( L(t,x) \). \([2]\) investigated the lag increments of \( L(t,x) \).

Motivated by these properties, we recall the corresponding lag increments (c. f. \([3, 8]\)). The aim of this paper is to get a uniform version of the lag increments almost sure asymptotic of \( Y(\cdot) \).

### § 2 Results Similar to Those on the Lag Increments of a Wiener Process

Throughout this paper, \( C \) will denote various positive constants, which may take different values at different places. We will use "log log x" to mean

\[ \log \log x = \log \log (\max(x, e)). \]

Sometimes \([x]\) will denote the greatest integer less than or equal to \( x \).

**Theorem 2.1.** We have

\[ \limsup_{T \to \infty} \sup_{0 \leq t \leq T} \frac{|Y(T) - Y(T - t)|}{d(T,t)} = 2, \quad \text{a.s.} \quad (2.1) \]

and

\[ \limsup_{T \to \infty} \sup_{0 \leq t \leq T} \sup_{h \leq T} \frac{|Y(s) - Y(s - h)|}{d(T,t)} = 2, \quad \text{a.s.} \quad (2.2) \]