Abstract. In this paper, the following result is given by using Hodge decomposition and weak reverse Hölder inequality: For every $r_1$ with
\[ p - \left( 2^{n+1} 100^n p \left( 2^{3+n/(p-1)} + 1 \right) \right)^{-1} < r_1 < p, \]
there exists the exponent $r_2 = r_2(n, r_1, p) > p$, such that for every very weak solution $u \in W^{1,\text{loc}}(\Omega)$ to A-harmonic equation, $u$ also belongs to $W^{1,\text{loc}}(\Omega)$. In particular, $u$ is the weak solution to A-harmonic equation in the usual sense.

§1 Introduction

Let $\Omega$ be a bounded regular domain in $\mathbb{R}^n (n \geq 2)$. By regular domain we understand any domain of finite measure for which the estimates for Hodge decomposition in (2.1) and (2.2) are justified. See [5], [7], [4]. A Lipschitz domain, for example, is regular.

We denote by $L^p(\Omega)$ ($1 \leq p < \infty$) the space of functions defined on $\Omega$, such that $|f(x)|^p$ is integrable with respect to the measure $dx$ with the norm
\[ \|f\|_{p,\Omega} = \left( \int_{\Omega} |f(x)|^p \, dx \right)^{1/p}. \]
The space $L^p_{\text{loc}}(\Omega)$ consists of functions which belong to $L^p(F)$ for every compact subset $F \subset \Omega$. The symbol $W^1_p(\Omega)$ ($W^1_p_{\text{loc}}(\Omega)$), $1 \leq p < \infty$ stands for the class of functions which belong to $L^p(\Omega)$ ($L^p_{\text{loc}}(\Omega)$) and whose weak partial derivatives exist and also belong to $L^p(\Omega)$ ($L^p_{\text{loc}}(\Omega)$).

We consider solutions $u$ of a quasilinear second order equation
\[ \text{div} A(x, \nabla u(x)) = 0, \quad (1.1) \]
where $A : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfies the usual measurability conditions (Caratheodory conditions) and that for some $1 < p < \infty$, the following conditions hold:

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(I) the Lipschitz type condition
\[ |A(x, \xi) - A(x, \zeta)| \leq b |\xi - \zeta| (|\xi| + |\zeta|)^{p-2}; \]

(II) the monotonicity inequality
\[ (A(x, \xi) - A(x, \zeta), \xi - \zeta) \geq a |\xi - \zeta|^2 (|\xi| + |\zeta|)^{p-2}; \]

(III) the homogeneity condition
\[ A(x, \lambda \xi) = |\lambda|^{p-2} \lambda A(x, \xi) \]
for almost every \( x \in \Omega \) and all \( \xi, \zeta \in \mathbb{R}^n \), \( 0 < a \leq b < \infty \), \( \lambda \in \mathbb{R} \).

**Remark 1.** The mapping \( A(x, \xi) = |\xi|^{p-2} \xi \), which generates the \( p \)-harmonic equation
\[ \text{div} \left( |\nabla u(x)|^{p-2} \nabla u(x) \right) = 0, \]

satisfies the assumptions (I) (II) (III).

We need the following definition.

**Definition 1.** A function \( u \in W^{1}_{p, \text{loc}}(\Omega) \) is said to be a weak solution to (1.1) if
\[ \int_{\Omega} \langle A(x, \nabla u), \nabla \phi \rangle \, dx = 0 \quad (1.2) \]
holds for all \( \phi \in W^{1}_{p, 0}(\Omega) \).

Recall that the original test function for weak solution is that \( \phi \in C_{0}^{\infty}(\Omega) \), and then by an approximation argument it is extended to all \( \phi \in W^{1}_{p, 0}(\Omega) \).

By the Lipschitz type inequality, it is clear that in order to guarantee the integrability of the integrand of (1.2) with \( \phi \in C_{0}^{\infty}(\Omega) \), it is only necessary that \( u \in W^{1}_{r, \text{loc}}(\Omega) \), where \( \max\{1, p-1\} \leq r \). For this reason we give the following definition.

**Definition 2.** A function \( u \in W^{1}_{r, \text{loc}}(\Omega) \) with \( \max\{1, p-1\} < r < p \) is called a very weak solution to (1.1), if (1.2) holds for all \( \phi \in W^{1}_{r, p-1, 0}(\Omega) \).

In [7], Iwaniec and Sbordone proved the following regularity result for very weak solutions to (1.1).

**Lemma 1.** There exist exponents
\[ 1 < r_{1} = r_{1}(n, p, a, b) < p < r_{2} = r_{2}(n, p, a, b) < \infty, \]
such that every very weak solution \( u \in W^{1}_{r_{1}, \text{loc}}(\Omega) \) to \( A \)-harmonic equation belongs to \( W^{1}_{r_{2}, \text{loc}}(\Omega) \).

**Lemma 2.** Let \( B_{R} \) denotes the open ball of radius \( R \). \( B_{\sigma R} \) stands for the ball of radius \( \sigma R \) with the same center as in \( B_{R} \). If \( u \in W^{1}_{p}(B_{R}), 1 \leq p < \infty \), then for any \( 0 < \sigma \leq 1 \),
\[ \| u - \bar{u}_{B_{\sigma R}} \|_{L^{p}(B_{R})} \leq \left( \frac{2}{\sigma} \right)^{n/p} \text{diam}B_{R} \| \nabla u \|_{L^{p}(B_{R})} \quad (1.3) \]
where \( \bar{u}_{B_{\sigma R}} = \frac{1}{|B_{\sigma R}|} \int_{B_{\sigma R}} u(x) \, dx \) is the average value of \( u(x) \) over \( B_{\sigma R} \).

Lemma 2 is from Lemma 1.5 in [1].