STUDY OF UPPER BOUND PROBLEM OF HEILBRONN TYPE

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Abstract. A set of \( n \) points in the plane determines a total \( C^n \) distances (some of them may be the same). Let \( r_n \) be the ratio of the maximum distance to the minimum distance, and \( R_n \) be the greatest lower bound for \( r_n \). By using the mathematical software Mathematica, the author gets the following results in this paper.

\[
R_{12} \leq 2.99496 \ldots , R_{13} \leq \csc \frac{\pi}{10}.
\]

The problem of Heilbronn type in combinatorial geometry is as the following:
A set of \( n \) points in the plane determines a total \( C^n \) distances (some of them may be the same). Let \( r_n \) be the ratio of the maximum distance to the minimum distance, and \( R_n \) be the greatest lower bound for \( r_n \). Find the value of \( R_n \) or give the bound of \( R_n \).

At present we know the following exact values about \( R_n \), namely\(^{(1-3)}\):

\[
R_3 = 1, R_4 = \sqrt{2}, R_5 = 2 \sin \frac{3\pi}{10}, R_6 = 2 \sin \frac{2\pi}{5}, R_7 = 2, R_8 = 1 + 2 \csc \frac{2\pi}{7}.
\]

On the estimation of \( R_n \), the author has proved\(^{(2-4)}\):

\[
c_1 \sqrt{n} - 1 \leq R_n \leq c_1 \sqrt{n}, \text{where} \ c_1 = \sqrt{\frac{12}{n}}.
\]

As for the upper and lower bound for \( R_n (n = 9, 10, 11, 12) \), we have:

\[
R_9 \leq R_{10} \leq 2.579, 2.32063 \ldots \leq R_{12} \leq 2.79377 \ldots ,
\]

\[
2.48271 \ldots \leq R_{11} \leq 2.90737 \ldots , 2.63756 \ldots \leq R_{12} \leq 2.9941 \ldots \text{ (see [5-6]).}
\]

In [6], the author has proved that \( R_{13} \leq 2 \sqrt{3} \). In this paper we want to give two new upper bounds for \( R_{12} \) and \( R_{13} \), i.e. \( R_{12} \leq 2.99496 \ldots , R_{13} \leq \csc \frac{\pi}{10} = 3.23606 \ldots \) According to [6], we have the following results;

\[
2.63756 \ldots \leq R_{12} \leq 2.99496 \ldots , 2.78099 \ldots \leq R_{13} \leq 3.23606 \ldots .
\]

In order to give the upper bound for \( R_{12} \), we will improve the 12-point-structure given
in [6]. As shown in Figure 1, we construct an equilateral triangle $MNP$ in the complex plane, with its unit length side and centered at the origin of coordinate $O$, point $M$ lies on y-axis. If we use lower case $m$, $n$, and $p$ to present the complex value of points $M$, $N$, and $P$, (the same meaning for the later use) we have:

$$m = \frac{\sqrt{3}}{3}i, n = m\omega, p = m\omega^2,$$

where $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$.

Let $\triangle AMI, \triangle CND, \triangle FPG$ be three congruent isosceles triangles, their vertices angles $2\alpha$ are not less than $60^\circ$, their waists are unit length.

The three congruent convex pentagons $ABCM, DEFP, GHIMP$ all have the unit length sides. We know that the positions of the remaining points $A, B, C, D, E, F, G, H, I$ are uniquely determined by angle $\alpha$. The six points $A, C, D, F, G, I$ can be represented as:

$$a = m + (-\sin\alpha + i\cos\alpha), d = a\omega, g = a\omega^2,$$

$$i' = m + (\sin\alpha + i\cos\alpha), c = k\omega, f = k\omega^2.$$

Let $h = x_0 + iy_0$, then the coordinate of point $H$ can be calculated accurately by mathematical software Mathematica as follows:

$$\text{sol} = \text{solve}\{ (x - \text{Re}[i])^2 + (y - \text{Im}[i])^2 = 1, (x - \text{Re}[g])^2 + (y - \text{Im}[g])^2 = 1 \}, \{x, y\}.$$

It has two solutions, in order to get convex pentagons, we chose greater solution for $x_0$.

$$x = \text{sol}[[2, 1, 2]]; y = \text{sol}[[2, 2, 2]].$$

In this way, we can find the coordinates for the three points $H, B, E; h = x_0 + iy_0, b = h\omega, e = h\omega^2$.

Due to $\alpha \geq 30^\circ$, we have $AI = CD = FG \geq 1$, so the minimum distance between these 12 points must be 1. For the reason of symmetry, we can see that the maximum distance between these 12 points must be the maximum of $AE, AF$ and $BH$.

Plot the graphs of $AE$ and $AF$ (They are functions of $\alpha!$) by using Mathematica, we have:

$$\text{Plot}\{\{\text{Abs}[a - e], \text{Abs}[a - f]\}, \{\alpha, 30 \, \text{Degree}, 40 \, \text{Degree}\}\}.$$