APP  OXIMATION OF CONVEX TYPE FUNCTION
BY PARTIAL SUMS OF FOURIER SERIES

Yu Guohua

Abstract. The concept of convex type function is introduced in this paper, from which a kind of
convex-decomposition approach is proposed. As one of applications of this approach, the
approximation of the convex type function by the partial sum of its Fourier series is inves-
tigated. Moreover, the order of approximation is described with the 2th continuous modulus.

§ 1 Introduction

Let \( f \in L[\pi, \pi] \) be a 2\( \pi \)-periodic function and its Fourier series
\[
f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)
\]
with following corresponding conjugate series;
\[
\sum_{n=1}^{\infty} (a_n \sin nx - b_n \cos nx).
\]

We denote by \( S_n(f, x) \) and \( \tilde{S}_n(f, x) \) the \( n \)th partial sums of the series (1.1) and
(1.2) respectively.

Approximation of a given function class by Fourier series has always been an
important topic in the function approximation theory. Many famous mathematicians have
made their contributions in this field. It is known that if \( f \) is a 2\( \pi \)-periodic continuous
function, i.e., \( f \in C^r[-\pi, \pi] \), then
\[
\| S_n(f, x) - f(x) \|_{C^r[-\pi, \pi]} \leq (3 + \log(n + 1))E_n^r(f).
\]
This result is the earliest work obtained by Lebesgue, and the following Theorem A holds:

Theorem A\[1\]. If \( f \in \text{Lip} \alpha \), \( 0 < \alpha \leq 1 \), then
\[
S_n(f, x) - f(x) = O\left( \frac{|\log n|}{n^\alpha} \right)
\]
holds uniformly with respect to \( x \).

Generally speaking, the \( \log n \) cannot be removed or reduced from (1.3) and (1.4) even if \( f \in \text{Lipa} \), \( 0 < a < 1 \). However, we are often interested in the subclass of \( C^*[-\pi, \pi] \). Salem and Zygmund succeeded in proving the following Theorem B.

**Theorem B**. If \( f \in \text{Lipa} \), \( 0 < a < 1 \), is a monotonic type function, then

\[
S_n(f,x) - f(x) = O(n^{-a})
\]

holds uniformly with respect to \( x \), where

\[
\bar{f}(x) = \lim_{\varepsilon \to 0} \frac{\varphi_x(f(x), \varepsilon)}{2 \tan(t/2)} dt \quad \text{and} \quad \varphi_x(f,x) = \frac{1}{2} [f(x + t) - f(x - t)].
\]

**Theorem C**. If \( f \in C^*[-\pi, \pi] \), is a monotonic type function, then

\[
S_n(f,x) - f(x) = O(n^{-1}) \sum_{k=1}^{\infty} \omega(f,k^{-1})
\]

holds uniformly with respect to \( x \), where \( \omega(f,x) \) is the ordinary continuous modulus of \( f \). Theorem C has also been generalized with the help of \( \omega \)-type monotonic function.

**Theorem D**. If \( f \in C^*[-\pi, \pi] \cap M^*[-\pi, \pi] \), then

\[
S_n(f,x) - f(x) = O(n^{-1}) \sum_{k=1}^{n} [\omega(f,k^{-1}) + \omega(k^{-1})]
\]

holds uniformly with respect to \( x \), where \( \omega(\delta) \) is a certain continuous modulus.

We see from the above that Theorem B, Theorem C and Theorem D request the function \( f \) to be of monotonic type. However, the monotonic type function is only a small subclass of bounded variation functions. So maybe we can remove the restricted condition, and therefore raise such a question: What is a subclass in the class of bounded variation functions \( BV^*[-\pi, \pi] \), which have more precise estimation of approximation by the partial sums of Fourier series? We find that the function belonging to this subclass possesses certain convexity. In this paper, we will discuss and answer this problem.

This paper is organized as follows. In § 2, we shall give the concept of the convex type function and discuss some properties of this type function. In § 3, we shall give the main results of this paper (see Theorem 1 and Theorem 2) and proofs. Also some direct consequences of Theorem 2 are stated. In § 4, we shall give the concept of \( \Phi \)-type convex function and then establish the corresponding Theorem 2 for this type function. Finally, an open problem is proposed.

## § 2 Preliminaries

We first recall the notion of convex function. Let \( I \) be an interval on a real line \( \mathbb{R} \) and