REMARKS ON CRITERIA OF PREQUASI-INVEX FUNCTIONS

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Abstract. Yang et al gave some criteria of prequasi-invex functions, semistrictly prequasi-invex functions and strictly prequasi-invex functions in 2001, under a certain set of conditions. In this note, some of these conditions can be weakened to get the same results, and another simplified proof for a criterion of prequasi-invex functions established under the condition of lower semicontinuity is given.

§ 1 Introduction

Convexity and generalized convexity play a central role in mathematical economics, engineering, management science, and optimization theory. Therefore, the research on convexity and generalized convexity is one of the most important aspects in mathematical programming. Recently, \cite{1} derived criteria for quasiconvex functions under lower semicontinuity and upper semicontinuity conditions. In \cite{2} Yang et al introduced various concepts of prequasi-invex functions, types of generalized version of quasiconvex functions. They established criteria of prequasi-invex functions, semistrictly prequasi-invex functions and strictly prequasi-invex functions under lower semicontinuity and upper semicontinuity conditions, thus generalizing the results in \cite{1}. In this paper, we show that some of their conditions can be weakened to get the same results. For example, we show that a prequasi-invex function is semistrictly prequasi-invex if there exists an $a \in (0,1)$ such that for every $x, y \in K$, $f(x) \neq f(y)$ implies that the following inequality (1) holds. Furthermore, we give another simplified proof for a criterion of prequasi-invex functions established under the condition of lower semicontinuity.

The following definitions are from \cite{2}.

\textbf{Definition 1.} For a given set $K \subseteq \mathbb{R}^n$ and a given function $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $K$ is said to be invex with respect to $\eta$ iff

\[ \forall x, y \in K, \forall \lambda \in [0,1] \Rightarrow y + \lambda \eta(x,y) \in K. \]
In this case, we also say that $K$ is an invex set with respect to $\eta$.

**Definition 2.** Let $K \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. Let $f : K \to \mathbb{R}$. We say that $f$ is prequasi-invex on $K$ iff $\forall \ x, y \in K, \forall \lambda \in [0, 1],$

$$f(y + \lambda \eta(x, y)) \leq \max\{f(x), f(y)\}.$$

**Definition 3.** Let $K \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $f : K \to \mathbb{R}$. We say that $f$ is strictly prequasi-invex on $K$ iff $\forall \ x, y \in K, x \neq y, \forall \lambda \in (0, 1),$

$$f(y + \lambda \eta(x, y)) < \max\{f(x), f(y)\}.$$

**Definition 4.** Let $K \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $f : K \to \mathbb{R}$. We say that $f$ is semistrietly prequasi-invex on $K$ iff $\forall \ x, y \in K, f(x) \neq f(y), \forall \lambda \in (0, 1),$

$$f(y + \lambda \eta(x, y)) < \max\{f(x), f(y)\}.$$

In [2] Yang et al discussed semicontinuity and prequasi-invexity of functions under the following conditions.

**Condition C.** Let $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. We say that the function $\eta$ satisfies Condition C iff $\forall \ x, y \in K, \lambda \in [0, 1],$

$$\eta(y, y + \lambda \eta(x, y)) = -\lambda \eta(x, y),$$

$$\eta(x, y + \lambda \eta(x, y)) = (1 - \lambda) \eta(x, y).$$

We see from the first equality of Condition C that $\eta(x, x) = 0, \forall \ x \in K$ because of taking $\lambda = 0$, and

$$\eta(y + \lambda \eta(x, y), y),$$

$$\eta(y + \lambda \eta(x, y), y + \lambda \eta(x, y) - \lambda \eta(x, y)) =$$

$$\eta(y + \lambda \eta(x, y), y + \lambda \eta(x, y) + \eta(y, y + \lambda \eta(x, y))) =$$

$$-\eta(y, y + \lambda \eta(x, y)) = \lambda \eta(x, y).$$

**Condition D.** Let the set $\Gamma$ be invex with respect to $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, and $f : \Gamma \to \mathbb{R}$. Then

$$f(y + \eta(x, y)) \leq f(x), \forall \ x, y \in \Gamma.$$

§ 2 Main result

In this paper, we always assume that

(i) $K \subseteq \mathbb{R}^n$ is an invex set with respect to $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n;

(ii) $\eta$ satisfies Condition C; $f$ is a real-valued function on $K$.

The following result was proved in [2] (see Theorem 3.1 in [2]).

**Theorem 1.** Let $f$ be a prequasi-invex function on $K$. Then, $f$ is a semistrietly prequasi-invex function on $K$ if and only if there exists an $\alpha \in (0, 1)$ such that for every $x, y \in K$, $f(x) \neq f(y)$ implies

$$f(y + \alpha \eta(x, y)) < \max\{f(x), f(y)\}, \quad (1)$$

$$f(y + (1 - \alpha) \eta(x, y)) < \max\{f(x), f(y)\}. \quad (2)$$

Now we improve the above theorem as follows.

**Theorem 2.** Let $f$ be a prequasi-invex function on $K$. If there exists an $\alpha \in (0, 1)$ such that