CENTER CONDITIONS AND BIFURCATION OF LIMIT CYCLES FOR A CLASS OF FIFTH DEGREE SYSTEMS

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Abstract. The center conditions and bifurcation of limit cycles for a class of fifth degree systems are investigated. Two recursive formulas to compute singular quantities at infinity and at the origin are given. The first nine singular point quantities at infinity and first seven singular point quantities at the origin for the system are given in order to get center conditions and study bifurcation of limit cycles. Two fifth degree systems are constructed. One allows the appearance of eight limit cycles in the neighborhood of infinity, which is the first example that a polynomial differential system bifurcates eight limit cycles at infinity. The other perturbs six limit cycles at the origin.

§ 1 Introduction

In the qualitative theory of planar differential equations, the center-focus determination and the bifurcation of limit cycles are known as two difficult problems. The computation of focal values is one way to study them. On the case of the origin of these problems, a lot of work have been done (see monograph \cite{1}). Most of them were investigated for cubic and quadratic systems. Recent papers \cite{2--5}, for example, concern the topics. But on the case of infinity, there are only a few results (see \cite{7--9}).

In \cite{8}, the following systems

\begin{align}
\frac{dx}{dt} & = -\eta y + p_2(x,y) + (-y + \delta x)(x^2 + y^2), \\
\frac{dy}{dt} & = \eta x + q_2(x,y) + (x + \delta y)(x^2 + y^2),
\end{align}

\begin{align}
(p_2(x,y), q_2(x,y)) & \text{ are quadratic homogeneous polynomials)}
\end{align}

have been considered. It is the first time that the focal values of the origin and the focal values of infinity are discussed in a class of systems. The authors computed the first five

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focal values at infinity and first four focal values at the origin, and discussed center conditions and bifurcations of limit cycles at the origin as well as at infinity. The computation methods they took were classic, i.e., the method of Poincaré return map and the method of Lyapunov coefficients. But using these methods, it is very complicated to compute focal values for $n$ degree systems ($n > 3$) and the results are not convenient to apply. Directly related with our work are [9] and [10]. In [10], the authors gave the definition of singular point quantity, in addition, put some computational methods that took the calculation of focal values and saddle qualities into calculation of singular point quantities. These are a class of new methods to compute focal values, which make it possible to compute the focal values of higher degree systems. The case of elementary critical point is discussed in [9], while the extended methods for the case of higher-critical point and infinity are presented in [9].

In this paper, we study a class of fifth degree systems with linear and cubic degree homogeneous polynomials

\[
\begin{align*}
\frac{dx}{dt} &= \lambda(-y + \delta_1x) + A_{30}x^3 + A_{21}x^2y + A_{12}xy^2 + A_{03}y^3 + (-y + \delta_2x)(x^2 + y^2)^2, \\
\frac{dy}{dt} &= \lambda(x + \delta_1y) + B_{30}x^3 + B_{21}x^2y + B_{12}xy^2 + B_{03}y^3 + (x + \delta_2y)(x^2 + y^2)^2.
\end{align*}
\]

(1.2)

where $\lambda, \delta_1, \delta_2, A_{30}, A_{21}, A_{12}, A_{03}, B_{30}, B_{21}, B_{12}, B_{03}$ are real constants, and $\lambda \neq 0$. In § 2, we outline some characters of system (1.2) and give the relation between focal values and singular point quantities. In § 3, we deduce a recursive formula for calculating singular point quantities at infinity of system (1.2) $|_{\delta_2 = 0}$. We calculate the first nine singular point quantities of infinity and get the conditions of infinity to be a center. The recursive formula we present in this section is linear and then avoids complex integrating operations. The calculation can be done readily with using computer algebra systems such as Mathematica or Maple. We discuss the case of the origin of system (1.2) $|_{\delta_2 = 0}$ in § 4.

In § 5 we construct two fifth degree systems belonging to system (1.2). One system allows the appearance of eight limit cycles in the neighborhood of infinity, which is, as far as we know, the first example that a polynomial differential system bifurcates eight limit cycles at infinity. The other system perturbs six limit cycles at the origin.

§ 2 Some preliminary results

System (1.2) has famous characteristics as follows. The origin is a center or a focus of the system, and the equator $\Gamma_0$ on the Poincaré closed sphere is a trajectory of the system, having no real critical point. $\Gamma_\infty$ is called the infinite point or infinity of system (1.2).

System (1.2) in polar coordinates, $x=r \cos \theta, y=r \sin \theta$, takes the form