MOMENT OF LEVY MEASURE OF OPERATOR-STABLE LAW

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Abstract. In 1979, Jurek gave a characterization of the moment of a full operator-stable \( \mu \) by eigenvalues of exponent matrix of \( \mu \). Here, a characterization of the moment of Lévy measure (restricted on a neighbor of 0) of a full operator-stable \( \mu \) by eigenvalues of exponent matrix of \( \mu \) is given.

§ 1 Introduction

Let \( \mathbb{R}^d \) denote the \( d \)-dimensional Euclidean space with inner product \( \langle \cdot, \cdot \rangle \) and the norm \( |\cdot| \). We write \( P(\mathbb{R}^d) \) for the set of probability measures on \( \mathbb{R}^d \), \( \mu \ast v \) for the convolution of \( \mu, v \in P(\mathbb{R}^d) \) and \( \delta_x (x \in \mathbb{R}^d) \) for the probability measure concentrated at the point \( x \). An element \( \mu \in P(\mathbb{R}^d) \) is called infinitely divisible if for any \( n = 2, 3, \ldots \), there exists \( \mu_n \in P(\mathbb{R}^d) \) such that \( \mu_n \ast \mu = \mu \). Further, \( \mu \) is infinitely divisible if and only if the characteristic function \( \hat{\mu} \) of \( \mu \) is of the norm

\[
\hat{\mu}(z) = \exp \left[ -\frac{1}{2} \langle z, Az \rangle + i \langle Y, z \rangle + \int_{\mathbb{R}^d} (e^{i\langle z, x \rangle} - 1 - i\langle z, x \rangle 1_D(x)) M(dx) \right].
\]

(1.1)

where \( D \) is the unit closed ball on \( \mathbb{R}^d \), \( A \) is a symmetric nonnegative-definite \( d \times d \) matrix, \( M \) is a measure on \( \mathbb{R}^d \) satisfying

\[
M(0) = 0 \text{ and } \int_{\mathbb{R}^d} (|x|^2 \wedge 1) M(dx) < \infty,
\]

(1.2)

and \( Y \in \mathbb{R}^d \) (refer to [5]). The representation (1.1) is unique and in the sequel we will write \( \mu = [A, M, Y] \) if \( \hat{\mu} \) has the representation (1.1).

A measure \( \mu \in P(\mathbb{R}^d) \) is said to be full if its support is not contained in any proper subspace of \( \mathbb{R}^d \). Given a linear operator \( B \) on \( \mathbb{R}^d \) and \( \mu \) in \( P(\mathbb{R}^d) \), we shall use \( B\mu \) to denote the probability measure defined by the formula \( (B\mu)(E) = \mu(B^{-1}(E)) \) for any Borel subset \( E \) on \( \mathbb{R}^d \).

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In paper [6], Sharpe defined and investigated the class of full operator-stable measures, i.e., the class of probability measure \( \mu \) on \( \mathbb{R}^d \) such that, for some probability measure \( \nu \) there exist nonsingular linear transformations \( \{ A_n \} \) and points \( \{ a_n \} \) such that the sequence \( \{ A_n \nu * \delta(a_n) \} \) converges weakly to \( \mu \). Sharpe also showed that a full measure \( \mu \) on \( \mathbb{R}^d \) is operator-stable if and only if
\[
\mu = t^\beta \mu * \delta(b(t)) \quad \text{for all } t > 0,
\]
where \( B \) is a nonsingular linear transformation, \( b : (0, \infty) \to \mathbb{R}^d \), and \( t^\beta \) is defined by the series
\[
\sum_{k=0}^{\infty} (\ln t)^k (k!)^{-1} B^k.
\]
Such a transformation \( B \) is called an exponent (matrix) for \( \mu \).

In the following, let \( \mu = [0, M, \gamma] \) be a full operator-stable measure on \( \mathbb{R}^d \) and \( B \) be an exponent for \( \mu \). Let \( h \) be the minimal polynomial of \( B \), and \( a_k, k = 1, 2, \ldots, d \), be the eigenvalues of \( B \), then from Theorem 3 and Theorem 4 of [6] we have
\[
\Re a_k > 1/2. \quad (1.3)
\]
Let \( \beta = \max_{1 \leq k \leq d} \Re a_k \). By Proposition 2.1 and Corollary 4.1 of [2] we get the following theorem.

**Theorem 1.1.** For a full operator-stable measure \( \mu = [0, M, \gamma] \) and \( r \geq 0 \), we have
\[
\int_{|x| \geq 1} |x|^r \mu(dx) < \infty \text{ if and only if } r < 1/\beta.
\]

Actually for any infinitely divisible measure \( \mu \), there exists a Lévy process \( X = (X_t, P^0) \) on \( \mathbb{R}^d \) such that \( \mu = P^0 * X_1 \). Let \( \pi_n : 0 = t_{s,1} < t_{s,2} < \ldots < t_{s,n} \leq 1 \) be a sequence of partitions of \( [0,1] \). The sequence \( \{ \pi_n \} \) will be called "nested" if \( \pi_{n+1} \) is a refinement of \( \pi_n \) for each \( n \geq 1 \). Define
\[
V(X,r,\pi_n) = \sum_{s \geq 1} |X(t_{s+1}) - X(t_s)|^r.
\]
As we know, the Lévy measure \( \mu \) manages the jumps of \( t \mapsto X_t \), and specifically, \( \mu \) restricted on \( \{ |x| \geq 1 \} \) manages the big jumps, while \( \mu \) restricted on \( \{ |x| < 1 \} \) manages the small jumps. The condition \( \int_{|x| \geq 1} |x|^r \mu(dx) < \infty \) if and only if \( E^0 |X_1|^r < \infty \). In this article, we give a characterization of the finiteness of moments of small jumps, \( \int_{|x| \leq 1} |x|^r \mu(dx) \), also in terms of the eigenvalues of the exponent matrix of \( \mu \). The problem is interesting since we know that if \( \{ \pi_n \} \) is a nested sequence of partitions of \( [0,1] \) and \( \int_{|x| \leq 1} |x|^r \mu(dx) < \infty \), then \( \sup_n V(X,r,\pi_n) < \infty \) almost surely. (See Theorem 3.1 of [3])

Let \( U \) denote the unit sphere in \( \mathbb{R}^d \). For any Borel subset \( E \) of \( \mathbb{R}^d \) \( \setminus \{0\} \) and any \( u \in U \), set \( M_u(E) := M_u(B) := \int_0^1 1_{\{t^\beta u \geq 1\}} t^{-2} dt \). We let \( L := L(B) := \{ u \in U : \text{for all } t, |t^\beta u| > 1 \} \) and set \( K(F) = M(t^\beta u \in F, t > 1) \), where \( F \subset L \) is of Borel subset, then
\[
M(E) = \int_L M_u(E)K(du) \quad (1.4)
\]
for any Borel subset \( E \) of \( \mathbb{R}^d \setminus \{0\} \). (See Theorem 2 of [1]).