A CLASS OF SINGULARLY PERTURBED NONLINEAR BOUNDARY VALUE PROBLEM

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Abstract. The singularly perturbed boundary value problem for the nonlinear boundary conditions is considered. Under suitable conditions, the asymptotic behavior of solution for the original problems is studied by using theory of differential inequalities.

§ 1 Introduction

The nonlinear singularly perturbed problem is a very attractive object of study in the international academic circles[1]. During the past decade many approximate methods have been developed and refined, including the method of averaging, boundary layer method, methods of matched asymptotic expansion and multiple scales. Recently, many scholars have done a great deal of work[2–7]. Using the method of differential inequality, Mo et al considered also a class of singularly perturbed nonlinear boundary value problems for the ordinary differential equation in Refs. [8, 9], the reaction diffusion equations in Refs. [10–12], the boundary value problems of elliptic equation in Refs. [13, 14], the initial boundary value problems of hyperbolic equation in Ref. [15], the shock layer solution of nonlinear equation for singularly perturbed problem in Refs. [16, 17] and the problems of biomathematics in Ref. [18]. In this paper, using a unique and simple method, we study a class of singularly perturbed nonlinear boundary value problem.

We consider the following nonlinear singularly perturbed problem:

\[ \varepsilon y''(t) = f(t, y(t), y'(t)), \ 0 \leq t \leq 1, \]  
\[ g(y(0), y'(0)) = 0, \]  
\[ h(y(1), y'(1)) = 0, \]

where \( \varepsilon \) is a positive small parameter.

Baxley studied a special problem[19] of (1)–(3).

An application concerning the stress boundary value problem for a symmetric
membrane is presented in [20].

We assume

\[ f,g \text{ and } h \text{ are sufficiently smooth bounded functions with regard to variables in corresponding ranges.} \]

[H2] There are constants \( M > 0, 0 \leq a_0 < b_0 \) and natural number \( n \), such that \( f(t, \cdot, x') \) is non-increasing for any \( (t, x') \in [0, 1] \times [a_0, b_0] \) and

\[ |f(t, y(t), y'(t))| \leq M |y'(t)|^{n+1}, a_0 \leq y'(t) \leq b_0, 0 \leq t \leq 1. \]

And there are positive constants \( k \) and \( k' \), \( k' \leq e_k \), such that \( f_y \leq -k, f_y \geq -k' \) in their variables.

[H3] \( g(y, \cdot), h(y, \cdot) \) and \( g(\cdot, y'), h(\cdot, y') \) are increasing for fixed \( y \) and \( y' \) respectively. And there are positive constants \( r_{ij} \), \( i, j = 1, 2 \), such that \( g_y \geq r_{ij}, g_y \leq r_{ij} \) and \( h_y \leq r_{21}, h_y \leq r_{22}, r_{ij} e^{-r_{ij} - r_{ij}^2} > 0, i = 1, 2 \), in their variables.

§ 2 Constructing formal asymptotic solution

We now construct the formal asymptotic expansions for solution of the problem (1)—(3).

The reduced problem of original problem is

\[ f(t, y(t), y'(t)) = 0, 0 \leq t \leq 1, \tag{4} \]
\[ h(y(1), y'(1)) = 0. \tag{5} \]

From (4) as \( t = 0 \), there is a solution \( s = s_1(r) \) of \( f(0, r, s) = 0 \). And from (5) we obtain \( r = r_1 \) which is the solution for \( h(r, s_1) = 0 \). Thus \( y(0) = r_1 \). Then we get a solution \( Y_0(t) \) for the problem (4), (5).

Now we construct the outer solution \( Y(t, \varepsilon) \) of the problem (1)—(3). Let

\[ Y(t, \varepsilon) = \sum_{\iota = 0}^{\infty} Y_{\iota}(t) \varepsilon^{\iota}. \tag{6} \]

Substituting (6) into (1), (3), developing \( f \) and \( h \) in \( \varepsilon \), incorporating and equating the terms of coefficients of like powers of \( \varepsilon \) for two sides of the equations respectively, we obtain

\[ f_y(t, Y_0, Y'_0) Y'_0 + f_x(t, Y_0, Y'_0) Y_0 = Y'_{\iota-1} + F_\iota, \iota = 1, 2, \ldots, \tag{7} \]
\[ h_y(Y_0(1), Y'_0(1)) Y'_0(1) + g_y(Y_0(1), Y'_0(1)) Y_0(1) = H_\iota, \iota = 1, 2, \ldots, \tag{8} \]

where \( F_\iota \) and \( H_\iota \) are determined functions. From linear problems (7), (8) we can obtain the solutions \( Y_{\iota}(t), \iota = 1, 2, \ldots, \) successively.

Substituting \( Y_{\iota}(t), \iota = 0, 1, 2, \ldots, \) into (6), we yield the outer solution \( Y(t, \varepsilon) \) of the problem (1)—(3). But it may not satisfy boundary condition (2), so we need construct for boundary layer the corrective term \( Z \) near \( t = 0 \).

We introduce a stretched variable \( \tau = t/\varepsilon \), and let

\[ u = Y(t, \varepsilon) + Z(\tau, \varepsilon), \tag{9} \]