EXISTENCE OF SOLUTIONS OF A FAMILY OF NONLINEAR BOUNDARY VALUE PROBLEMS IN $L^2$-SPACES

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Abstract. By using the perturbation results of sums of ranges of accretive mappings of Calvert and Gupta (1978), the abstract results on the existence of solutions of a family of nonlinear boundary value problems in $L^2(\Omega)$ are studied. The equation discussed in this paper and the methods used here are extension and complement to the corresponding results of Wei Li and He Zhen's previous papers. Especially, some new techniques are used in this paper.

§ 1 Introduction and preliminaries

The equations related to the $p$-Laplacian operator $\Delta_p$ have been studied in different aspects since $\Delta_p$ arises from a variety of physical phenomena such as non-Newtonian fluids, reaction-diffusion problems, etc. In this paper, the following equation (1) which can be regarded as an extension of those in [1-4] will be discussed:

$$
\begin{cases}
- \text{div}(a(\text{grad} u)) + |u(x)|^{p-2}u(x) + g(x,u(x)) = f(x) & \text{a.e. on } \Omega \\
- \langle n, a(\text{grad} u) \rangle \in \beta_p(u(x)), & \text{a.e. on } \Gamma.
\end{cases}
$$

(1)

More details can be seen in § 2. And, the result that a family of equations related to $p$ in (1) have solutions in the same space $L^2(\Omega)$, where $\frac{2N}{N+1} < p < +\infty$ and $N \geq 1$, will be obtained.

Now, let $X$ be a real Banach space with a strictly convex dual space $X^*$. We use "w-lim" to denote the weak convergence. For any subset $G$ of $X$, we denote by int$G$ its interior and $\overline{G}$ its closure, respectively. Let $X \hookrightarrow Y$ indicate that space $X$ is embedded compactly in space $Y$. A mapping $T : D(T) = X \rightarrow X^*$ is said to be hemi-continuous on $X$ if w-lim$_{t \to 0} T(x-ty) = Tx$, for any $x, y \in X$. Let $J$ denote the duality mapping from $X$ into $2^X$ defined by $J(x) = \{ f \in X^* : \langle x, f \rangle = \| x \| \cdot \| f \|, \| f \| = \| x \| \}, \forall x \in X$, where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $X$ and $X^*$. Since $X^*$ is strictly convex, so $J$
is a single-valued mapping.

Let \( A : X \to 2^X \) be a given multi-valued mapping. We say that \( A \) is boundedly-inversely-compact if for any pair of bounded subsets \( G \) and \( G' \) of \( X \), the subset \( G \cap A^{-1}(G') \) is relatively compact in \( X \). The mapping \( A : X \to 2^X \) is said to be accretive if \( (v_1 - v_2, J(u_1 - u_2)) \geq 0 \) for any \( u_i \in D(A) \) and \( v_i \in Au_i, i = 1, 2 \). The accretive mapping \( A \) is said to be \( m \)-accretive if \( R(I + \lambda A) = X \) for some \( \lambda > 0 \).

Let \( B : X \to 2^{X^*} \) be a given multi-valued operator. The graph of \( B, G(B) \), is defined by \( G(B) = \{ [u, w] | u \in D(B), w \in Bu \} \). Then \( B : X \to 2^{X^*} \) is said to be monotone if \( G(B) \) is a monotone subset of \( X \times X^* \) in the sense that \( (u_i - u_2, w_1 - w_2) \geq 0 \) for any \( [u_i, w_i] \in G(B), i = 1, 2 \). The operator \( B \) is said to be maximal monotone if \( G(B) \) is monotone and is maximal among all monotone subsets of \( X \times X^* \) in the sense of inclusion. \( B \) is said to be coercive if \( \lim_{n \to +\infty} \frac{\langle x_n, x_n^* \rangle}{\| x_n \|^2} = +\infty \) for all \( [x_n, x_n^*] \in G(B) \) such that \( \lim_{n \to +\infty} \| x_n \| = +\infty \).

We need the following theorem introduced by Calvert and Gupta in paper [5], which demonstrates how to use the perturbation theory for maximal monotone operators to discuss the existence of solution of nonlinear elliptic boundary value problem.

**Definition 1.1.** [5] The duality mapping \( J : X \to X^* \) is said to be satisfying Condition (I) if there exists a function \( \eta : X \to [0, +\infty) \) such that \( \forall u, v \in X, \| J_u - J_v \| \leq \eta(u - v) \).

**Definition 1.2.** [5] Let \( A : X \to 2^X \) be an accretive mapping and \( J : X \to X^* \) be a duality mapping. We say that \( A \) satisfies Condition (\( * \)) if for every \( f \in R(A) \) and \( a \in D(A) \), there exists a constant \( C(a, f) \) such that for \( u \in D(A), v \in Au \), we have

\[
(v - f, J(u - a)) \geq C(a, f).
\]

**Theorem 1.1.** [5] Let \( X \) be a real Banach space with a strictly convex dual \( X^* \). Let \( J : X \to X^* \) be a duality mapping on \( X \) satisfying Condition (I). Let \( A, B_1 : X \to 2^X \) be accretive mappings such that

(i) either both \( A, B_1 \) satisfy Condition (\( * \)) or \( D(A) \subseteq D(B_1) \) and \( B_1 \) satisfies Condition (\( * \)),

(ii) \( A + B_1 \) is \( m \)-accretive and boundedly-inversely-compact.

Let \( B_2 : X \to X \) be a bounded continuous mapping such that for any \( y \in X \), there is a constant \( C(y) \) satisfying \( (B_2(u + y), Ju) \leq -C(y) \) for \( \forall u \in X \). Then

(a) \( [R(A) + R(B_1)] \subseteq R(A + B_1 + B_2) \),

(b) \( \text{int}[R(A) + R(B_1)] \subseteq \text{int}R(A + B_1 + B_2) \).

**§ 2 Main results**

Let \( \Omega \) be a bounded conical domain of a Euclidean space \( \mathbb{R}^N (N \geq 1) \) with its boundary \( \Gamma \in C^1 \) and suppose that Green’s Formula is available. Let \( \| \cdot \| \) denote the Euclidean norm in \( \mathbb{R}^N, \langle \cdot, \cdot \rangle \) the Euclidean inner-product and \( n \) the exterior normal derivative of \( \Gamma \).