Some results on P-harmonic maps and exponentially harmonic maps between Finsler manifolds

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Abstract. This paper studies the stability of P-harmonic maps and exponentially harmonic maps from Finsler manifolds to Riemannian manifolds by an extrinsic average variational method in the calculus of variations. It generalizes Li’s results in [2] and [3].

§1 Introduction

Harmonic maps between Finsler manifolds are defined as the critical points of the energy functionals. A harmonic map is said to be stable if its second variation of the energy functional is always nonnegative; otherwise unstable. In [6], Shen and Zhang have derived the first and second variational formulas of the energy functional for a nondegenerate map between Finsler manifolds. In particular, they obtained the nonexistence of nonconstant stable harmonic maps from a compact Finsler manifolds to the standard unit spheres $S^n$, $n > 2$. In [5], Shen and Wei have extend the results of [6] from the standard spheres to a class of manifolds with positive Ricci curvature, known as superstrongly unstable manifolds.

In [2] and [3], Li has derived the first and second variation formulas for P-harmonic maps and exponentially harmonic maps between Finsler manifolds respectively. The following two results were established.

Theorem 1.1. There is no nondegenerate stable P-harmonic map between a Riemannian unit sphere $S^n$ for $n > P \geq 2$ and any compact Finsler manifold.

Theorem 1.2. There is no nondegenerate stable exponentially harmonic map $\phi$ between a Riemannian unit sphere $S^n$ and any compact Finsler manifold with $|d\phi|^2 < n - 2$.

The main purpose of this paper is to generalize partially the above two theorems in the following two theorems.

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Theorem 1.3. Let $\tilde{M}^m$ be a compact submanifold of the Euclidean space $R^{m+q}$ with flat normal bundle. Let $H^\alpha = (h^\alpha_{ij})_{m \times m}$ for $\alpha = m + 1, \ldots, m + q$, where $h$ is the second fundamental form of $\tilde{M}^m$ in $R^{m+q}$. If $H^\alpha$ is positive definite for every $\alpha$ and at every point on $\tilde{M}^m$ and the principal curvature satisfy
\[ 1 \leq \frac{\lambda^\alpha_{\max}}{\lambda^\alpha_{\min}} < \sqrt{\frac{m}{P}}, \quad (P < m), \quad \alpha = m + 1, \ldots, m + q, \]
where $\lambda^\alpha_{\max} = \max\{\lambda^\alpha_1, \ldots, \lambda^\alpha_m\}$ and $\lambda^\alpha_{\min} = \min\{\lambda^\alpha_1, \ldots, \lambda^\alpha_m\}$, then there is no nondegenerate stable $P$-harmonic map from any Finsler manifold $M$ to such Riemannian manifold $\tilde{M}^m$ for $m > P \geq 2$.

Theorem 1.4. Let $\tilde{M}^m$ be a compact submanifold of the Euclidean space $R^{m+q}$ with flat normal bundle and $|d\phi|^2 \leq C$ for some positive number $C$. Let $H^\alpha = (h^\alpha_{ij})_{m \times m}$ with $\alpha = m + 1, \ldots, m + q$, where $h$ is the second fundamental form of $\tilde{M}^m$ in $R^{m+q}$. If $H^\alpha$ is positive definite for every $\alpha$ and at every point on $\tilde{M}^m$, the principal curvatures satisfy
\[ 1 \leq \frac{\lambda^\alpha_{\max}}{\lambda^\alpha_{\min}} < \frac{m}{C + 2}, \quad \alpha = m + 1, \ldots, m + q, \]
where $\lambda^\alpha_{\max} = \max\{\lambda^\alpha_1, \ldots, \lambda^\alpha_m\}$ and $\lambda^\alpha_{\min} = \min\{\lambda^\alpha_1, \ldots, \lambda^\alpha_m\}$, then there is no nondegenerate stable exponentially harmonic map from any Finsler manifold $M$ to such Riemannian manifold $\tilde{M}^m$.

Moreover, we will give some examples.

\section{Preliminaries}

Let $M$ be an $n$-dimensional smooth manifold and $\pi : TM \to M$ be the natural projection from the tangent bundle $TM$. Let $(x, y)$ be a point of $TM$ with $x \in M$, $y \in T_x M$, and let $(x^i, y^i)$ be the local coordinates on $TM$ with $y = y^i \frac{\partial}{\partial x^i}$. A Finsler metric on $M$ is a function $F : TM \to [0, +\infty)$ satisfying the following properties.

(i) Regularity: $F(x, y)$ is smooth in $TM \setminus 0$.
(ii) Positive homogeneity: $F(x, \lambda y) = \lambda F(x, y)$ for $\lambda > 0$.
(iii) Strong convexity: The fundamental quadratic form
\[ g = g_{ij}(x, y)dx^i \otimes dx^j, \quad g_{ij} = \frac{1}{2}[F^2]_{y^i y^j} \]
is positively definite.

Here and from now on, $[F]_{y^i}$ stands for $\frac{\partial F}{\partial y^i}$ and $[F]_{y^i y^j}$ stands for $\frac{\partial^2 F}{\partial y^i \partial y^j}$, etc.; and we use the following convention of index ranges unless otherwise stated:
\[ 1 \leq i, j, k, \ldots \leq n, \quad 1 \leq a, b, c, \ldots \leq m. \]

Since the $g_{ij}$'s are homogeneous of degree zero in $y$, they are well-defined quantities on the projective sphere bundle $SM$ with respect to the $y$'s as homogeneous coordinates. The canonical projection $\pi : TM \to M$ gives rise to a covector bundle $\pi^* T^* M$ which has a global section $\omega = [F]_{y^i} dx^i$ called the Hilbert form, whose dual vector field is
\[ l = l^i \frac{\partial}{\partial x^i} \quad \text{with} \quad l^i = \frac{y^i}{F}. \]