$L^p$ estimates for the Schrödinger type operators

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Abstract. Let $L_k = (-\Delta)^k + V^k$ be a Schrödinger type operator, where $k \geq 1$ is a positive integer and $V$ is a nonnegative polynomial. We obtain the $L^p$ estimates for the operators $\nabla^2 L_k^{-1}$ and $\nabla^k L_k^{-\frac{1}{2}}$.

§1 Introduction

In this paper we are concerned with the Schrödinger type operator

$$L_k = (-\Delta)^k + V^k \quad \text{on } \mathbb{R}^n, \quad n > 2k$$

where $k \geq 1$ is a positive integer and $V(x)$ is a nonnegative polynomial. This operator can be regarded as a natural generalization of Schrödinger operator $L_1 = -\Delta + V$. Recently, some scholars have paid close attention to this operator (cf. [1,2,12]). In particular, when $k = 2$, Zhong obtained the $L^p$ boundedness of the operator $\nabla^4 L_2^{-1}$ in [13]. Furthermore, Sugano [11] obtained the same result as Zhong’s under a weaker condition on $V$. The $L^p$ estimates of the operator $\nabla^2 L_2^{-\frac{1}{2}}$ had been studied by Liu and Dong in [8] for $1 < p \leq p_0$, where the nonnegative potential $V$ belongs to reverse Hölder class $B_q$ and $\frac{1}{p_0} = \frac{2}{q} - \frac{2}{n}$. The purpose of this paper is to establish the $L^p$ boundedness for the operators $\nabla^{2k} L_k^{-1}$ and $\nabla^k L_k^{-\frac{1}{2}}$. Throughout the paper we only consider the operators $L_k$ for $k \geq 3$. The main reason is that the operators $L_1$ and $L_2$ have been investigated in [9], [13] and [8]. From now on, we always denote the Schrödinger type operator $L_k$ simply by $L$.

To state our main results, we first recall some definitions and notations. A nonnegative locally $L^q$ integrable function $V$ on $\mathbb{R}^n$ is said to belong to reverse Hölder class $B_q$ ($1 < q < \infty$) if there exists a constant $C > 0$ such that the reverse Hölder inequality

$$\left( \frac{1}{|B|} \int_B V(x)^q \, dx \right)^\frac{1}{q} \leq C \left( \frac{1}{|B|} \int_B V(x) \, dx \right)$$

(1)

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holds for every ball $B$ in $\mathbb{R}^n$. It follows from (0.12) in [9] that if $V$ is a nonnegative polynomial, then $V$ belongs to $B_q$ for all $q$, $1 < q < \infty$.

Let $\alpha = (\alpha_1, \cdots, \alpha_n)$ denote the multi-index with $\alpha_i \in \mathbb{N}$. Define $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $\partial^\alpha = D^\alpha = \partial^{\alpha_1}/\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}$ for $|\alpha| = \alpha_1 + \cdots + \alpha_n$. For any positive integer $k$ and a smooth function $u$, denote $D^k u(x) = (\partial^\alpha u(x) : |\alpha| = k)$ and $|D^k u(x)|^2 = \sum_{|\alpha|=k} |\partial^\alpha u(x)|^2$. Also, for any smooth functions $u$ and $v$, denote $D^k u(x) D^j v(x) = \sum_{|\alpha|=k, |\beta|=j} C_{\alpha, \beta} \partial^\alpha u(x) \partial^\beta v(x)$.

Let $H_t(x, y) = e^{-t(-\Delta)^k}(x, y)$ be the polyharmonic heat kernel of $(-\Delta)^k$ in $\mathbb{R}^n$. It follows from [4] that there exist positive constants $c_1$ and $c_2$ such that

$$ |H_t(x, y)| \leq c_1 t^{-\frac{n}{2}} \exp\{-c_2 \frac{|x-y|^2}{t}\}. \quad (2) $$

Different from the case of Laplacian, the polyharmonic heat kernel $H_t(x, y)$ is no longer positive (cf. [5] and [7]). The Schrödinger type operator $L$ generates a $(C_0)$ semigroup $\{T^s_t : s > 0\} = \{e^{-sL} : s > 0\}$. Let $K^L_s(x, y)$ denote the kernel of $T^s_t$. Because the kernel $K^L_s(x, y)$ is also not positive, we can’t use Trotter product formula to obtain the estimate in (2) as the case of $k = 1$. But Proposition 5.2 in [3] implies that the following upper bounds of the kernel $K^L_s(x, y)$ are valid, that is,

$$ |K^L_s(x, y)| \leq c_1 t^{-\frac{n}{2}} \exp\{-c_2 \frac{|x-y|^2}{t}\}. \quad (3) $$

Assume $V \in B_\infty^p$. We recall the auxiliary function $\rho(x)$ introduced by Shen [9] defined by

$$ \rho(x) = \frac{1}{m(x, V)} = \sup_{r>0} \left\{ r : \frac{1}{r^n-1} \int_{B(x, r)} V(y) \, dy \leq 1 \right\}, \quad x \in \mathbb{R}^n, $$

where $B(x, r)$ denotes the ball with centre $x$, radius $r$. The properties of the auxiliary function $m(x, V)$ are given by Shen [9]. It is known that $0 < \rho(x) < \infty$ for any $x \in \mathbb{R}^n$ (from Lemma 1.2 in [9]). If $V$ is a polynomial with the degree of $l$, it follows from [9, p.517] that $m(x, V) \sim \sum_{|\alpha| \leq l} |\partial^\alpha V(x)|^{l+1}$.

We have the following results.

**Theorem 1.1.** Suppose $V$ is a nonnegative polynomial. Then for $1 < p \leq \infty$, there exists a constant $C > 0$ such that

$$ ||V^k L^{-1} f||_{L^p(\mathbb{R}^n)} \leq C \| f \|_{L^p(\mathbb{R}^n)} . \quad (4) $$

**Corollary 1.1.** Suppose $V$ is a nonnegative polynomial. Then there exists a positive constant $C$ such that, for $1 < p < \infty$,

$$ ||\nabla^{2k} L^{-1} f||_{L^p(\mathbb{R}^n)} \leq C \| f \|_{L^p(\mathbb{R}^n)}, \quad (5) $$

where $\nabla^{2k} = \nabla^2 = \partial^{\alpha_1}/\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}$, $2k = |\alpha| = \alpha_1 + \cdots + \alpha_n$.

**Theorem 1.2.** Suppose $V$ is a nonnegative polynomial. Then for $1 < p < \infty$, there exists a constant $C > 0$ such that

$$ ||\nabla^k L^{-\frac{1}{2}} f||_{L^p(\mathbb{R}^n)} \leq C \| f \|_{L^p(\mathbb{R}^n)}, $$

where $\nabla^k = \nabla^k = \partial^{\alpha_1}/\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}$, $k = |\alpha| = \alpha_1 + \cdots + \alpha_n$. 