INVARlANT MEASURE, RATIO LIMITS
AND MARTIN BOUNDARY

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Abstract. In this article the notion of quasi-symmetry is introduced. It is proved that the quasi-
symmetry is equivalent to the uniqueness of invariant measure of Lévy processes in some sense.
Moreover, the relationship between ratio limits and invariant measures is studied.

§ 1 Introduction

Let $X=(X_t;P^x)$ be a Lévy process on $\mathbb{R}^n$ with the convolution semigroup $\{\pi_t; t>0\}$. For any probability measure $\mu$ on $\mathbb{R}^n$, define the characteristic function of $\mu$ on $\mathbb{R}^n$ as $\hat{\mu}(u) := \int e^{i\langle u, y \rangle} \mu(dy)$. Then $\pi = \{\pi_t\}$ is a convolution semigroup with Lévy exponent $\phi$, i.e. $\pi_t(u) = e^{-\phi(t)}$.

For any probability measure $\mu$ on $\mathbb{R}^n$, we define the moment generating function $\mathcal{L}$ of $\mu$ on $\mathbb{R}^n$ as $\mathcal{L}_\mu(u) := \int e^{i\langle u, y \rangle} \mu(dy)$. Let $\varphi = \mathcal{L}_\pi$, then $\mathcal{L}_\pi = \varphi$. The Martin boundary of $\pi$ is defined by $E := \{u: \varphi(u) = 1\}$.

For any $u \in E$, the measure $e^{i\mathcal{L}_u}dx$ is an invariant measure of $X$.

Let $\beta(\mathbb{R}^n)$ denote the Borel $\sigma$-algebra of $\mathbb{R}^n$ and let $\beta(\mathbb{R}^n)^c := \{A \in \beta(\mathbb{R}^n); \text{the closure of } A \text{ is compact}\}$. Let $m, \cdot \cdot$ and $dx$ denote the Lebesgue measure. For any $u \in \mathbb{R}^n$, let $\|u\|$ denote the length of $u$.

We call $X=(X_t;P^x)$ non-singular if for some $t>0$, $\pi_t$ has a non-trivial absolutely continuous (w. r. t. Lebesgue measure) component. Otherwise, the process is called singular.

A $\sigma$-finite measure $\mu$ on $\mathbb{R}^n$ is called an invariant measure for $X$ if $\mu \ast \pi_t = \mu$ for all $t>0$. Denote by $\text{Inv}$ the set of all invariant measure for $X$. Clearly $m \in \text{Inv}$. We say that $X$ has a unique invariant measure if $\mu \in \text{Inv}$ implies that $\mu$ is a multiple of $m$ and that $X$ has a

Received: 2001-09-24.
MR Subject Classification: 60J27, 60J45.
Keywords: convolution semigroup, invariant measure, Martin boundary, quasi-symmetry.
unique Radon invariant measure if the only Radon measure in Inv are multiples of $m$.

First we find that the Condition 1 which appears in [2] and [3] is very interesting and important. Thus in § 2 we introduce the concept of quasi-symmetry to describe the Condition 1 and give some equivalent conditions. We also prove that a quasi-symmetric Lévy process has a unique Radon invariant measure if and only if its Lévy exponent has a unique zero at 0. In 1966 Ney and Spitzer[11] gave a description of the Martin boundary for a random walk. In § 3, from a result of Port and Stone, we obtain a ratio limit theorem for non-singular Lévy processes.

§ 2 Quasi-symmetric and invariant measure

In this section, we shall make the basic assumption that $\pi_1$ is not supported on a proper subspace of $\mathbb{R}^n$. Since in general cases there exists a subspace $H$ of $\mathbb{R}^n$ such that when regarding $\{\pi_1; t > 0\}$ as a convolution semigroup on $H$ it satisfies the assumption. Our assumption involves no loss of generality.

In 1971 Port and Stone[2] proved some ratio limit theorems. These theorems take their nicest form when $X$ satisfies

**Condition 1.** There exists a compact subset $K \subseteq \mathbb{R}^n$ such that

$$\limsup_{t \to \infty} \sup \pi_t(K)^{1/2} = 1.$$  

**Definition 2.1.** We say that $\{\pi_1; t > 0\}$ is quasi-symmetric if it satisfies Condition 1. We say that $X$ is quasi-symmetric if its corresponding convolution semigroup $\{\pi_1; t > 0\}$ is quasi-symmetric.

We say that $\{\pi_1; t > 0\}$ is degenerate if for some $a \in \mathbb{R}, u \in \mathbb{R}^n$ with $u \neq 0, \pi_1$ is supported on $\{x: (u, x) = a\}$. Otherwise $\{\pi_1; t > 0\}$ is said to be nondegenerate.

Let $F = \{u \psi(u) < \infty\}$, we have the following lemma.

**Lemma 2.1.** $\psi$ is a convex extended real-valued function and $F$ is a convex set. Furthermore, $\psi$ is a strictly convex function of $F$.

**Proof.** Since $x \mapsto e^x$ is a strictly convex function of $\mathbb{R}, \psi$ is a convex function and $F$ is a convex subset. By way of contradiction, suppose that $\psi$ is not a strictly convex function on $F$, then there exist $u_1, u_2 \in F, u_1 \neq u_2$ such that $\psi(au_1 + (1 - a)u_2) = a\psi(u_1) + (1 - a)\psi(u_2)$ for some $0 < a < 1$. Since $e^x$ is strictly convex, $(u_1, x) = (u_2, x)$ a.e. $\pi_1$. Hence $\sup \pi_1 \subseteq \{x: (u_1 - u_2, x) = 0\}$ which contradicts the basic assumption.

**Lemma 2.2.** Any local minimum point of $\psi$ is a global minimal point and there is only one local minimum point of $\psi$.

**Proof.** By way of contradiction, suppose a minimum point $u_0$ is local but not global. Then there exist $u \neq u_0$ such that $\psi(u) < \psi(u_0)$ and $\psi(au + (1 - a)u_0) \leqslant a\psi(u) + (1 - a)\psi(u_0) < \psi(u_0)$ for all $0 < a < 1$. This contradicts that $u_0$ is a local minimum point. So $u_0$ is a global minimal point.