SOLVABILITY OF A QUATERNION MATRIX EQUATION

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Abstract. This paper discusses the solvability of quaternion matrix equation $A^* X^* B^* + B X A = D$ and obtains its general explicit solutions in terms of $A, B, D$ and their Moore-Penrose inverses.

§ 1 Introduction

Let $R$ be the real number field, $C = R \oplus Ri$ be the complex number field, and $H = C \oplus Cj = R \oplus Ri \oplus Rj \oplus Rk$ be the quaternion division ring over $R$, where $k_1 = ij = ji$, $i^2 = j^2 = k^2 = -1$. If $a = a_1 + a_2i + a_3j + a_4k \in H$, where $a_i \in R$, then let $\overline{a} = a_1 - a_2i - a_3j - a_4k$ be the conjugate of $a$. Let $H^{m \times n}$ be the set of all $m \times n$ matrices over $H$. If $A = (a_{ij}) \in H^{m \times n}$, let $A^T$ be the transpose matrix of $A$, $\overline{A}$ be the conjugate matrix of $A$, and $A^* = (\overline{a}_{\overline{ij}})^T$ be the transpose conjugate matrix of $A$. $A \in H^{m \times n}$ is said to be Hermite matrix if $A = A^*$, and to be skew-Hermite matrix if $A = -A^*$.

The Moore-Penrose inverse of matrix $A \in H^{m \times n}$, denoted by $A^+$, is a matrix $X \in H^{n \times m}$ which satisfies these equations

$$AXA = A, \quad XAX = X, \quad (AX)^* = AX, \quad (XA)^* = XA.$$  

As in [4, 5], for $A \in H^{m \times n}$, $A$ can be uniquely written as

$$A = A_1 + A_2j, \quad \text{where} \quad A_1, A_2 \in C^{m \times n},$$  

and we define the $2m \times 2n$ complex matrix $A_c$ by

$$A_c = \begin{pmatrix} \overline{A_1} & A_2 \\ -\overline{A_2} & \overline{A_1} \end{pmatrix}.$$  

$A_c$ is called the complex representation matrix of $A$. By the properties of complex representation matrix (cf. Lemma 1 in [5]), it is easy to see that $(A_c)^+ = (A^+)$, and $(A_c)^* = (A^*)$, where $(A_c)^+$ is the Moore-Penrose inverse of $A$, according to the usual definition in complex matrix theory. Thus the Moore-Penrose inverse of $A$ is unique and the followings properties hold:

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\[
A = AA^* (A^+)^* = (A^+)^* A^* A, \\
A^+ = A^+ (A^+) A^* = A^* (A^+) A^+ , \\
A^* = A^+ AA^* = A^* AA^+, \\
(A^*)^+ = (A^+)^*, \quad A = (A^+)^+, \\
(A^* A)^+ = A^+ (A^*)^+, \\
AA^+ = (AA^+)^+ = (AA^*)^*, \quad A^+ A = (A^+ A)^+ = (A^+ A)^*. 
\]

For example, by properties of complex representation matrix and Moore-Penrose inverse of complex matrix, we have

\[
A_x = A x (A_x^+)^* = (A x (A_x^+)^*) x, \quad \text{and} \quad A_x = (A^+)^* A_x^* A = ((A^+)^* A^* A)^*, 
\]
thus (1.4) holds. Similarly, we can prove (1.5)–(1.9).

For \( A \in H^{n \times n} \), let

\[
E_A = I - AA^+, \quad F_A = I - A^+ A , 
\]
then we have

\[
E_A = E_A^* = E_A^+, \quad F_A = F_A^* = F_A^+. 
\]

The symmetric solution of the linear matrix equation over the real and complex number field has been investigated in \([1-3]\) etc.

Recently, the matrix equation over a ring or division ring has been widely studied (cf. \([5-8]\]).

A Hermite matrix \( X \) is called a Hermite solution of a matrix equation if \( X \) is a solution of this matrix equation.

In this paper we discuss the Hermite solution of the quaternion matrix equation

\[
AXB = D 
\]
and solvability of the quaternion matrix equation

\[
A^* X^* B^* \pm B X A = D. 
\]

In § 2, we give necessary and sufficient conditions for Eq. (1.12) to have a Hermite or skew-Hermite solution as well as explicit formula in terms of \( A, B, D \) and their Moore-Penrose inverses. In § 3, we consider the solvability of the matrix Eq. (1.13). We obtain necessary and sufficient conditions for the existence of solutions and their general forms in terms of \( A, B, D \) and their Moore-Penrose inverses. Compared with other discussions of the similar problem over \( C \) or \( H \), our results are easy to use in theory and can be generalized to other cases.

\section*{§ 2 Matrix equation \( A X B = D \)}

\textbf{Lemma 2.1.} \([4]\) Let \( A \in H^{m \times n}, B \in H^{p \times q}, D \in H^{n \times q} \), then the quaternion matrix equation

\[
AXB = D 
\]
has a solution if and only if

\[
AA^+ DB^+ B = D. 
\]