A NEW APPROACH TO SOLVE PERTURBED NONLINEAR EVOLUTION EQUATIONS THROUGH LIE-BÁCKLUND SYMMETRY METHOD

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Abstract. A new method based on Lie-Bäcklund symmetry method to solve the perturbed nonlinear evolution equations is presented. New approximate solutions of perturbed nonlinear evolution equations stemming from the exact solutions of unperturbed equations are obtained. This method is a generalization of Burde's Lie point symmetry technique.

§ 1 Introduction

Lie group theory gives a very general and effective approach to solve the nonlinear partial differential equations, which is different from other methods because it can reduce the original equations\(^{[6,7]}\). In some circumstances, the small perturbations are added to many classical equations which then become perturbed nonlinear equations. In recent years, there has been much interest in finding the solutions of these perturbed equations, and many methods to solve perturbed equations have appeared. For example, in \([1]\) the direct perturbation method is used to solve a fifth-order singularly perturbed KdV equation; in \([2]\) qualitative analysis is used to show that the existence of wavefront of perturbed Fisher equation with a fourth-order spatial derivative perturbation; in \([3]\) Jacobian elliptic function method is used to obtain exact solutions of perturbed KdV equation; in \([4]\) Burde applied Lie point symmetry technique for the perturbed nonlinear wave equation. In our paper, we apply Lie-Bäcklund symmetry method and generalize Burde's method. Because the order of the \(k\)th extended infinitesimal generator is not higher than \(k\) order when we use Lie point transformations to \(k\)-order partial differential equations, the equations to be solved must be restricted within those equations whose order of perturbation are not higher than the order of unperturbed equations. But in Lie-Bäcklund symmetry method there are not these restrictions. Through Lie-Bäcklund symmetry we also can obtain solutions of perturbed nonlinear evolutionary equations,
especially the equations whose perturbations are more complicated. In fact, the mapping relation of approximate solutions of perturbed equations and exact solutions of unperturbed equations is established by this method. The effect that perturbation produces can be studied.

§ 2 A generalization of Burde's Lie point symmetry technique

We will consider a $k$th-order nonlinear evolutionary partial differential equation depending on a small parameter $\varepsilon$, namely,

$$
\Delta(x, u, u^{(1)}, u^{(2)}, \ldots, u^{(k)}; \varepsilon) = \Delta_0(x, u, u^{(1)}, u^{(2)}, \ldots, u^{(k)}; \varepsilon) + \varepsilon \Delta_1(x, u, u^{(1)}, u^{(2)}, \ldots, u^{(k)}) = 0. \quad (1)
$$

We use the following notation: $x = \{x_1, x_2, \ldots, x_p\}$ denotes $p$ independent variables, $u$ denotes the dependent variable and $u^{(k)}$ denotes the set of all $k$th-order partial derivatives of $u$ with respect to $x$.

Let $\varepsilon = 0$, and the counterpart of this perturbed equation is the unperturbed equation:

$$
\Delta_0(x, u, u^{(1)}, u^{(2)}, \ldots, u^{(k)}) = 0. \quad (2)
$$

The main points of our generalized method are as follows.

(a) We consider the Lie-Bäcklund transformations of the following form:

$$
X(u, u^{(1)}, u^{(2)}, \ldots, u^{(m)}) = \eta(x, u, u^{(1)}, u^{(2)}, \ldots, u^{(m)}), \quad (3)
$$

where $m$ is a non-negative integer, $\varepsilon \ll 1$ and the infinitesimal generator of (3) is

$$
X = \eta(x, u, u^{(1)}, u^{(2)}, \ldots, u^{(m)}) \frac{\partial}{\partial u^*}. \quad (4)
$$

In these transformations, the independent variables $x$ are invariant and $\eta$ depends on the $m$th-order partial derivatives of $u$ with respect to $x$. These transformations are different from Lie point transformations used in Burde's paper [4].

The above transformations are applied to the unperturbed equation written in variables $x^*, u^*$ as

$$
\Delta_0(x^*, u^*, u^{(1)}_*, u^{(2)}_*, \ldots, u^{(k)}_*) = 0, \quad (5)
$$

which is transformed to

$$
\Delta_0(x^*, u^*, u^{(1)}_*, u^{(2)}_*, \ldots, u^{(k)}_*; \varepsilon) = 0
$$

or infinitesimally

$$
\Delta_0(x^*, u^*, u^{(1)}_*, u^{(2)}_*, \ldots, u^{(k)}_*) = \Delta_0(x, u, u^{(1)}, u^{(2)}, \ldots, u^{(k)}) + \varepsilon X^{(k)} \Delta_0(x, u, u^{(1)}, u^{(2)}, \ldots, u^{(k)})|_{\varepsilon=0} + O(\varepsilon^2). \quad (6)
$$

where $X^{(k)}$ is $k$th extended infinitesimal generator of (4), and $O(\varepsilon^2)$ denotes terms of order $\varepsilon^2$, $k \geq k$ is a non-negative integer.

(b) If

$$
X^{(k)} \Delta_0(x, u, u^{(1)}, u^{(2)}, \ldots, u^{(k)})|_{\varepsilon=0} = \Delta_1(x, u, u^{(1)}, u^{(2)}, \ldots, u^{(k)}),
$$