TWO GENERALIZATIONS OF
HARDY-LITTLEWOOD MAXIMAL OPERATOR

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Abstract. Two generalizations of Hardy-Littlewood maximal operator are considered. Some estimates for them are obtained.

§ 1 Introduction

Hardy-Littlewood maximal operator has wide applications in many fields, such as quasiconformal analysis, partial differential equations (PDEs) and harmonic analysis. Let \( \Omega \) be an open subset of \( \mathbb{R}^n \), the Hardy-Littlewood maximal operator is defined on \( L^p(\Omega) \) by the rule

\[
Mh(x) = M_\Omega h(x) = \sup \left\{ \frac{1}{|B(x,t)|} \int_B |h(y)| \, dy : 0 < t < \text{dist}(x, \partial \Omega) \right\},
\]

where, as usual, \( \int_B |h| = \frac{1}{|B|} \int_B |h| \) stands for the mean value of \( |h| \) over the set \( B \). In the above definition, \( B = B(x,t) \) is the ball centered at \( x \in \Omega \) with radius \( t \). Most often the dependence of \( M \) on the domain \( \Omega \) will not be emphasized.

We record the following local variant of the well-known maximal inequality

\[
\| Mh \|_{L^p(\Omega)} \leq C_p(n) \| h \|_{L^p(\Omega)}
\]

for all \( h \in L^p(\Omega), 1 < p \leq \infty \), where \( C_p(n) \leq \frac{PC(n)}{p-1} \), \( C(n) \) is some constant depending only on the dimension \( n \). Another important maximal operator is the following spherical maximal operator which involves spherical averages and was introduced to harmonic analysis in [1],

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\[ Sh(x) = S_{\partial}h(x) = \sup \left\{ \int_{S(x,t)} |h(y)| \, dy : 0 < t < \text{dist}(x, \partial \Omega) \right\}. \] (3)

Here and also in what follows we use the notation \( S(x,t) = \partial B(x,t) \). Notice that the integral average of \( |h| \) is taken with respect to the \((n-1)\)-dimensional surface area. As shown in [2] for \( n=2 \) and in [1] in higher dimensions, the spherical maximal operator is bounded in \( L^p \)-spaces for all \( p > n/(n-1) \), but not for \( p = n/(n-1) \). That is

\[ \| Sh \|_{L^p(\partial \Omega)} \leq C_p(n) \| h \|_{L^p(\partial \Omega)}, p > n/(n-1). \] (4)

T. Iwaniec and J. Onninen\(^{[3]} \) introduced a one-parameter family \( \{ M_{\theta} \}_{\theta \geq 1} \) of maximal operators

\[ M_{\theta}h(x) = \sup \left\{ \left( \frac{n}{\theta} \int_0^{\frac{n}{\theta}} r^{n-1} \left( \int_{S(x,r)} |h(y)| \, dy \right)^\theta \, dr \right)^{\frac{1}{\theta}} : 0 < t < \text{dist}(x, \partial \Omega) \right\}. \] (5)

and proved that the sublinear operator \( M_\theta : L^p(\Omega) \rightarrow L^p(\Omega) \) is bounded for all \( p > \frac{n}{n-1 + \frac{1}{\theta}} \).

They employed the spherical maximal functions successfully in the study of Jacobians for the first time.

In this note, we consider two generalizations of Hardy-Littlewood operator and obtain some estimates for them. We first generalize Iwaniec and Onninen's maximal operators to weak maximal operators and prove that they are also bounded from \( L^p(\Omega) \) to \( L^p(\Omega) \) for all \( p > \frac{n}{n-1 + \frac{1}{\theta}} \). Then we define the Morrey type operators and show that they are closely related to the cubical \( \alpha \)-dimensional outer Hausdorff measure by using Vitali Covering Lemma.

\section*{§ 2 Weak maximal operators}

We shall now introduce the following two-parameter family \( \{ W_{\alpha}M_{\theta} \}_{\theta \geq 1, 0 < \alpha < 1} \) of weak maximal operators

\[ W_{\alpha}M_{\theta}h(x) = \sup \left\{ \left( \frac{n}{\theta} \int_0^{\frac{n}{\theta}} r^{n-1} \left( \int_{S(x,r)} |h(y)| \, dy \right)^\theta \, dr \right)^{\frac{1}{\theta}} : 0 < t < \text{dist}(x, \partial \Omega) \right\}. \] (6)

Obviously, \( W_1M_{\theta}h(x) = M_{\theta}h(x) \) and the Hardy-Littlewood operator is none other than \( W_1M_1 \), whereas the spherical operator arises by letting \( \theta \) go to infinity. We obviously have

\[ W_{\alpha}M_{\theta}(\alpha h(x)) = |\alpha| W_{\alpha}M_{\theta}(h(x)), \forall \alpha \in \mathbb{R}, \]

\[ W_{\alpha}M_{\theta}(h(x) + g(x)) \leq W_{\alpha}M_{\theta}(h(x)) + W_{\alpha}M_{\theta}(g(x)). \]

This shows that the operator \( W_{\alpha}M_{\theta} \) is sublinear. In consequence of the maximal inequalities in (2) and (3) we have the following result.

**Theorem 1.** The sublinear operator \( W_{\alpha}M_{\theta} : L^p(\Omega) \rightarrow L^p(\Omega) \) is bounded for all \( p > \frac{n}{n-1 + \frac{1}{\theta}} \).