A NOTE ON CITY BLOCK DISTANCE

DING REN, XU CHANGQING AND LI YINGZI

Abstract. In this paper, problem of characterizing the city block distance between two lattice points in k-dimensional Euclidean space is discussed.

§ 1 Introduction

One of the most frequently used distances in the analysis of shape for pattern recognition, as generally speaking, in digital geometry, is the city block distance.

[1] gives some characterizations of city block distance. One of them arises from the so-called condition R and condition T.

To clarify the problems, we must first list the related notations and definitions.

We denote by \( E^k \) the k-dimensional Euclidean space, by \( Z^k \) the set of points in \( E^k \) with integer coordinates.

Definition 1. The city block distance between the points \( P(p_1, p_2, \ldots, p_k) \) and \( Q(q_1, q_2, \ldots, q_k) \) in \( E^k \) is given by

\[
d(P,Q) = \sum_{j=1}^{k} |p_j - q_j|.
\]

Definition 2. A metric \( d \) defined on \( Z^k \) satisfies condition R (Rosenfeld) if for all \( P(p_1, p_2, \ldots, p_k), Q(q_1, q_2, \ldots, q_k) \) in \( Z^k \) with \( P \neq Q \) the following conditions hold:

1) \( d(P,Q) \geq 1 \),
2) \( d(P,Q) = 1 \) iff there exists an integer \( j, 1 \leq j \leq k \), such that \( p_i = q_i \) for all \( i \neq j \) and \( |p_j - q_j| = 1 \).

Definition 3. A metric \( d \) on an arbitrary metric space \( X \) satisfies condition T if \( d(P,Q) = 1 \) implies that for all \( R \in X, |d(P,R) - d(Q,R)| = 1 \).

A theorem in [1] claims that the only metric on \( Z^k \) which satisfies both condition R and condition T is the city block metric. But unfortunately there is a gap in the proofs and the claim needs some adjustment.

In this notes, we first present a counterexample to show that there exists a distance...
defined on $\mathbb{Z}^d$ other than the city block distance, while it satisfies both condition R and condition T. Then we give a characterization of the city block distance after making an adjustment in the definition of condition T.

§ 2 A Counterexample

First we construct a metric on $\mathbb{Z}^d$, which satisfies both condition R and condition T, but is not the city block metric.

For $P(p_1, p_2, \ldots, p_k)$ and $Q(q_1, q_2, \ldots, q_k)$ in $\mathbb{Z}^k$, let

$$d(P, Q) = \begin{cases} 0, & P = Q, \\ 1, & \sum_{i=1}^k |p_i - q_i| = 1, \\ 2, & \sum_{i=1}^k |p_i - q_i| = 2, 4, 6, \ldots, \\ 3, & \sum_{i=1}^k |p_i - q_i| = 3, 5, 7, \ldots. \end{cases}$$

Proposition 1. The function $\tilde{d}$ defined by (a) is a metric on $\mathbb{Z}^d$.

Proof. From the definition of $\tilde{d}$, it is clear that for each pair of points $P$ and $Q$ in $\mathbb{Z}^d$

(i) $d(P, Q) \geq 0$, (ii) $d(P, Q) = 0$ iff $P = Q$, (iii) $d(P, Q) = d(Q, P)$.

Now we need only show that $d$ on $\mathbb{Z}^d$ satisfies the triangle inequality, i.e.

(iv) $d(P, Q) \leq d(P, R) + d(R, Q)$ for all $P, Q, R \in \mathbb{Z}^d$.

We have three cases to consider.

Case 1. $P = Q$. In this case, $d(P, Q) = 0$, and (iv) holds trivially.

Case 2. $P \neq Q$, but $R = P$ or $R = Q$. In this case the equality in (iv) holds.

Case 3. $P \neq Q$, $P \neq R$ and $Q \neq R$. In this case $d(P, R) \geq 1$ and $d(Q, R) \geq 1$.

Hence $\tilde{d}(P, R) + \tilde{d}(Q, R) \geq 2$.

So, if $d(P, Q) = 1$ or 2, we obtain $\tilde{d}(P, Q) \leq d(P, R) + d(Q, R)$.

It is left for us to verify that if $d(P, Q) = 3$, then $\tilde{d}(P, R)$ or $\tilde{d}(Q, R)$ is no less than 2.

Suppose $\tilde{d}(P, R) < 2$ and $\tilde{d}(Q, R) < 2$, then, from $\tilde{d}(P, R) \geq 1$ and $\tilde{d}(Q, R) \geq 1$, we get

$$\tilde{d}(P, R) = 1, \quad \tilde{d}(Q, R) = 1.$$

i.e.

$$\sum_{i=1}^k |p_i - r_i| = 1 \quad \text{and} \quad \sum_{i=1}^k |q_i - r_i| = 1,$$

thus there exist $l$ and $m(1 \leq l, m \leq k)$ such that

$$|p_i - r_i| = 1, \quad p_i = r_i \quad \text{when} \quad i \neq l;$$

$$|q_m - r_m| = 1, \quad q_m = r \quad \text{when} \quad i \neq m.$$