FINITE ELEMENT APPROXIMATIONS OF TWO-PHASE MISCIBLE INCOMPRESSIBLE DISPLACEMENT WITH DISCONTINUOUS COEFFICIENTS

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Abstract. Two-phase, miscible, incompressible flow in porous media is governed by a system of nonlinear partial differential equations. Many numerical methods have been given by different authors to this system, but these methods need very high regularity conditions. Actually, in most practical applications these regularity conditions couldn't be satisfied. In this paper, the problem of discontinuous coefficients with lower regularity conditions is considered and the error estimates are demonstrated.

§ 1 Introduction

Miscible displacement of one incompressible fluid by another in a porous medium $\Omega \subseteq \mathbb{R}^2$ of unit thickness and nonuniform local elevation can be described by a differential system (in which case gravity terms to be neglected in this presentation so as to concentrate attention on more essential portions of the problems should be included).

\begin{align*}
\nabla \cdot u &= -\nabla \cdot (a(x,c) \nabla p) = q_+ - q_- , \quad (1.1) \\
\phi(x) \frac{\partial c}{\partial t} + \nabla \cdot (uc) - \nabla \cdot D \nabla c &= q_+ \dot{c} - q_- c , \quad (1.2)
\end{align*}

for $x \in \Omega$ and $t \in J=[0,T]$, where $a(x,c) = k(x)/\mu(c)$ with $k(x)$ the permeability of the porous medium and $\mu(c)$ the concentration-dependent viscosity of the fluid mixture, $p$ is the pressure, $\phi$ the porosity, $u$ the Darcy velocity of the fluid, $c$ the concentration of one of the basic components in the mixture. The functions $q_+ \geq 0$ and $q_- \geq 0$ represent the strength of sources and sinks in the domain. The function $\dot{c}$ specifies the composition of the injected fluid. The diffusion coefficient $D=D(x,u)$ is a $2 \times 2$ matrix given by

\begin{equation}
D = \phi(x) \{ d_{mI} + |u| (d_{mE}(u) + d_{vE}(u)) \}, \quad (1.3)
\end{equation}
where \( E \) represents projection along the velocity vector and is given by
\[
e_{ij}(u) = u_i u_j / |u|^2,
\]
and \( E^\perp = I - E \). The diffusion coefficient \( d_{\text{long}} \) measures the dispersion in the direction of the flow, and \( d_{\text{trans}} \) that is transverse to the flow.

In practice, the coefficients \( k(x) \) and \( \phi(x) \) usually depend on the properties of the media that form the petroleum reservoir \( \Omega \). Generally, \( \phi(x) \) and \( k(x) \) have different values in particular subregion \( \Omega_s \subset \Omega \), \( s = 1, \ldots, m \), made from different media. Hence, \( \phi(x) \) and \( k(x) \) are discontinuous across the common boundaries of \( \Omega_s \), \( s = 1, \ldots, m \), where instead of system (1.1), (1.2), the so-called transition conditions are used.

In this paper we present a general theory of the finite element solution to system (1.1), (1.2) with discontinuous coefficients in a bounded domain \( \Omega \subset \mathbb{R}^2 \). We generalize here the methods and techniques from [1] where the finite element discretization of elliptic problem was studied, and [2] where the Galerkin methods for miscible displacement problems in porous media were studied.

\section{Assumptions}

Let \( \Omega \subset \Omega_1, \ldots, \Omega_m \subset \mathbb{R}^2 \) be bounded domains with Lipschitz continuous boundaries \( \partial \Omega \), \( \partial \Omega_1, \ldots, \partial \Omega_m \) and let
\[
\bar{\Omega} = \bigcup_{r=1}^m \Omega_r, \Omega_r \cap \Omega_s = \emptyset \quad \text{for } r, s = 1, \ldots, m, r \neq s,
\]
\[
\Omega_0 = \bigcup_{r=1}^m \Omega_r.
\]

We set
\[
\Gamma_n = \Gamma_n = \partial \Omega \cap \partial \Omega_r, r, s = 1, \ldots, m, r \neq s
\]
\[
\Gamma_{\text{ad}} = \partial \Omega \cap \partial \Omega_s, s = 1, \ldots, m,
\]
and let \( \partial \Omega, \Gamma_n \) to be formed by a finite number of open arcs or simple closed curves.

Obviously
\[
\partial \Omega = \Gamma_n \cup \left( \bigcup_{r=1}^m \Gamma_{\text{ad}} \right), \partial \Omega = \bigcup_{r=1}^m \Gamma_n.
\]

Of course, some of the sets \( \Gamma_n, \Gamma_{\text{ad}} \) can be empty.

If \( \Omega \rightarrow \mathbb{R}^1 \), then by \( g' \) we denote an extension of \( g \) onto \( \tilde{\Omega} \). Let \( \tilde{n'}(x) = (n'_1(x), n'_2(x)) \) denote the unit outer normal to \( \partial \Omega \). Obviously \( \tilde{n'}(x) = -n'(x) \), for \( x \in \Gamma_n \). We will consider the following differential system:
\[
\nabla u = - \nabla \cdot (a(x,c) \nabla p) = q(x,t), x \in \Omega, t \in J,
\]
\[
\phi(x) \frac{\partial c}{\partial t} + \nabla \cdot (uc) - \nabla \cdot D \nabla c = f(x,t,c), x \in \Omega, t \in J,
\]
with the boundary conditions
\[
u' \cdot \tilde{n'}(x) = 0,
\]