Finite time convergent control using
terminal sliding mode

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Abstract: A method for terminal sliding mode control design is discussed. As we know, one of the strong points of terminal sliding mode control is its finite-time convergence to a given equilibrium of the system under consideration, which may be useful in specific applications. The proposed method, different from many existing terminal sliding mode control design methods, is studied, and then feedback laws are designed for a class of nonlinear systems, along with illustrative examples.

Keywords: Terminal sliding mode control; Finite-time convergence; Nonlinear systems

1 Introduction

Variable structure system control, as one of the most active research areas of control theory and one of the powerful practical tools, has been studied for many decades. Sliding mode controllers are constructed in order to keep controlled systems to given constraint surfaces and also make the systems insensitive to some certain external and internal disturbances. Many results on sliding mode control and its extensions can be found in the literature including [1–4].

Recent years have also witnessed an increasing interest in terminal sliding mode control. This may be due to the fact that this non-smooth feedback may possess faster convergent rates (related to finite-time convergence) and superior robustness properties in practice. Terminal sliding mode control approach is to render the closed-loop system finite-time to converge in finite time to the desired position of the system under consideration, rather than only to a sliding surface. In this way, the dynamic response of the closed-loop system may be improved. Theoretical results and their application to robotic systems can be found in the references like [5–8], though the control methods may cause singularity problems. [5] proposed a two-phase control scheme to avoid the singularity of their original control law. Also, [7] proposed another idea to construct the discontinuous sliding modes with finite time convergence. In addition, it is worthwhile to point out that, besides discontinuous terminal sliding mode control, continuous finite-time control draws much research attention as well [9–12].

The aim of the paper is to give a new approach to terminal sliding mode controller, which may remove singularities outside of sliding surfaces. The rest of the paper is organized as follows. In Section 2, the problem formulation is given, while in Section 3 theoretical results to construct sliding mode controllers and discussion on terminal sliding mode control are shown. Then, in Section 4, terminal sliding mode control laws are built for a class of nonlinear systems. Finally, concluding remarks are given in Section 5.

2 Problem formulation

Consider the nonlinear control system

\[ \dot{x} = f(x) + g(x)u, \quad u \in \mathbb{R}, \quad x \in \mathbb{R}^n, \]

Let \( s : \mathbb{R}^n \rightarrow \mathbb{R} \) be a continuous function and the sliding surface is defined as \( S = \{ x \in \mathbb{R}^n : s(x) = 0 \} \). The corresponding motion, satisfying \( s(x) < 0 \), is called a sliding dynamics with respect to the constraint function \( s \). Conventionally, an ideal sliding motion on the sliding mode is described as

\[ \begin{align*}
    s(x) &= 0, \\
    \dot{s}(x) &= Lf + g\delta(x) = 0.
\end{align*} \]

In other words, \( S \) should be invariant if possible feedback, called an equivalent control, can be constructed.

The basic idea of sliding mode control is as follows. Choose a sliding manifold; then use the sliding mode control to drive the state outside of the manifold into the manifold; finally, using \( u_{eq} \) to render the state in the sliding mode along the plane to the desired equilibrium. Therefore, the stability problem basically changes to a problem how to find a suitable sliding mode and a sliding mode controller.

In this paper, we focus on a single-input control system of the form:

\[ \begin{align*}
    \dot{x}_1 &= f_1(x), \\
    \vdots \\
    \dot{x}_{n-1} &= f_{n-1}(x), \\
    \dot{x}_n &= f_n(x) + u,
\end{align*} \]
with \( x = (x_1, \cdots, x_n)^T \in \mathbb{R}^n \), \( f_i(x) \) smooth, and \( f_i(0) = 0 \) for \( i = 1, \cdots, n \).

The sliding mode design procedure can be divided roughly into two main steps:

1) Find a sliding surface satisfying transverse condition, that is, there is an \( i \) such that
\[
\frac{\partial s}{\partial x_i} \neq 0, 
\]
(without loss of generality, we take \( i = n \) in the sequel) and that \( s(x) = 0 \) implies \( x \rightarrow 0 \);

2) Construct a stabilizing feedback law in the form of
\[
u(x) = \begin{cases} u_m(x), & \text{if } S = 0, \\ u^+(x), & \text{if } s > 0, \\ u^-(x), & \text{if } s < 0. \end{cases}
\]

Most sliding mode control laws usually make the controlled system (1) converge to their sliding surfaces in finite time, and then, along the sliding surfaces, the systems converges to the equilibrium \( x = 0 \) of system (1) as time goes to infinity. However, terminal sliding mode control laws achieve more by steering the states to reach the equilibrium in finite time.

**Definition 1** Consider a system
\[
\dot{x} = f(x) + g(x)u, \quad x \in \mathbb{R}^n, 
\]
where \( f(0) = 0 \) and \( g(0) \neq 0 \). \( u = \mu(x) \) is called a finite-time convergent controller if the equilibrium \( x = 0 \) of the closed-loop system (3) under this feedback law is finite-time convergent, namely, for any initial condition \( x(0) = x^0 \in \mathbb{R}^n \), there is a settling time \( T \geq 0 \), which satisfies:
\[
\lim_{t \to T} x(t;0,x^0) = 0, 
\]
and \( x(t;0,x^0) = 0, \) if \( t > T \) for every solution \( x(t;0,x^0) \) to the closed-loop system (3). Moreover, if the controller is also a sliding mode controller, it is called a terminal sliding mode controller.

In what follows, for simplicity, we assume that \( s \) is selected in the following form to satisfy the transverse condition (2):
\[
s(x) = x_n - h(x_1, \cdots, x_{n-1}), \quad h(0, \cdots, 0) = 0, 
\]
where \( h(x_1, \cdots, x_{n-1}) \) is a continuous function. Thus, \( s(x) \) is obtained once \( h(x) \) is fixed. Here we give an assumption for the selection of \( h(x) \), that is,
\[
|h(x_1, \cdots, x_{n-1})| < K_0 + L_0 \sum_{i=1}^{n-1} |x_i| 
\]
for suitable positive numbers \( K_0 \) and \( L_0 \).

Many choices of sliding modes satisfy condition (4). For example, \( s(x) = \sum_{i=1}^n a_i x_i \) is a widely used form, and taking \( K_0 = 1 \) and \( L_0 = \max |a_i|, i = 1, \cdots, n-1 \) will make (4) hold.

**3 Main results**

At first, two lemmas are introduced, whose proofs are quite obvious and therefore omitted here.

**Lemma 1** For any \( 0 < \alpha < 1 \) and \( M_0 > 0 \), there is a \( M_1 > 0 \) such that \( |z|^\alpha \leq M_0 + M_1 |z| \) holds for all \( z \in \mathbb{R} \).

**Lemma 2** Suppose that \( a, b \), and \( m > 1 \) are all positive numbers. Then \( (a^m + b^n)^{1/m} \leq a + b \).

Denote \( \text{sgn} (\cdot) \) as the sign function. Then we have a result related to the convergence to sliding surfaces for system (1):

**Theorem 1** If the sliding surface \( S \) is taken with \( h(x) \) satisfying (4), then the control law
\[
u = -f_n(x) + v, 
\]
where
\[
v = -K(1 + \sum_{i=1}^{n-1} |f_i(x)|) \text{sgn} (s), 
\]
\[K > \max \{L_0, 1\}, \quad s \neq 0, 
\]
will ensure that the system (1) reaches \( S \) in finite time.

**Proof** The task, in fact, is to prove that, for any initial condition \( x(0) \neq 0 \) with \( s(x(0)) \neq 0 \), feedback (5) can make \( s(x(t)) = 0 \) in finite time.

Let \( x(0) = x^0 = (x_1^0, \cdots, x_{n-1}^0, x_n^0) \) be the initial condition. The trajectory with the initial condition is denoted by \( x(t;0,x(0)) \), or \( x(t) \) for simplicity.

We first study the case when \( s(x(0)) > 0 \), namely, \( x_n^0 > h(x_1^0, \cdots, x_{n-1}^0) \). We will prove that there is \( T > 0 \) such that \( s(x(T)) = 0 \) by contradiction.

Suppose that there is not any \( T > 0 \) such that \( s(x(T)) = 0 \). Because \( s(x(0)) > 0 \) and \( s(x(t)) \) is continuous, \( s(x(t)) > 0 \) for any \( t > 0 \), namely, \( x_n(t) > h(x_1(t), \cdots, x_{n-1}(t)) \) for any \( t > 0 \). Therefore,
\[
x_n = -K(1 + \sum_{i=1}^{n-1} |f_i(x)|) \text{sgn} (s) = -K(1 + \sum_{i=1}^{n-1} |f_i(x)|). 
\]
Integrating both sides of (7) gives
\[
x_n(t) - x_n^0 = -K t - K \sum_{i=1}^{n-1} \int_0^t |f_i(x)| \, dt 
\]
\[
\leq -K t - K \sum_{i=1}^{n-1} \int_0^t x_i \, dt 
\]
\[
\leq -K t - K \sum_{i=1}^{n-1} x_i(t) + K \sum_{i=1}^{n-1} x_i^0 
\]
which implies
\[
x_n(t) \leq -(K t - x_n^0 - K \sum_{i=1}^{n-1} x_i^0) = -K \sum_{i=1}^{n-1} x_i(t) / K. 
\]
Recall condition (4), i.e.,
\[
|h(x_1, \cdots, x_{n-1})| < K_0 + L_0 \sum_{i=1}^{n-1} |x_i|, 
\]
we have
\[
t > \frac{K_0 + \sum_{i=1}^{n-1} x_i^0}{K} + \sum_{i=1}^{n-1} |x_i^0|, 
\]
and therefore, we obtain the desired result.

At last, three examples are given to illustrate the effectiveness of the proposed controller.