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Rational cohomology of the free loop space of a simply connected 4-manifold

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Abstract. The purpose of this paper is to calculate the rational cohomology $H^*(X^{S^1};\mathbb{Q})$ of the free loop space for a simply connected closed 4-manifold X. We use minimal models, so the starting point is the cohomology algebra $H^*(X;\mathbb{Q})$ which depends only on the second Betti number b_2 and the signature of X itself. Calculations of $H^*(X^{S^1};\mathbb{Q})$ for $b_2 \leq 2$ are known. We study the case $b_2 > 2$. We obtain an explicit formula for Poincaré series of the space X^{S^1}, with the second Betti number b_2 as a parameter.

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1. Introduction

Let X be a simply connected closed manifold. Denote by X^{S^1} the space of all free loops in X.

The cohomology of the space X^{S^1} has been calculated for some particular spaces X. Spheres, complex projective spaces and spaces with cohomology with one or two generators were considered in [6, 12].

In the present work, $H^*(X^{S^1},\mathbb{Q})$ is calculated for simply connected closed 4-manifolds with second Betti number b_2 greater than 2. Let us note that calculations for $b_2 \leq 2$ can be easily obtained from results of [6, 12].

From now on we consider (co)homology with coefficients in $\mathbb{Q}$, unless otherwise stated.

Our main tool is Sullivan minimal model for a fibration. A minimal model of a simply connected space X is a free graded commutative algebra ΛV over $\mathbb{Q}$ with differential d satisfying certain conditions. Its definition will be given later; we just note that it is unique in a natural way and its cohomology $H^*(\Lambda V, d)$ is isomorphic to $H^*(X,\mathbb{Q})$. Here $V = \bigoplus_{i>0} V_i$ is a graded vector space; ΛV is the free graded commutative algebra generated

by V, that is tensor product of a polynomial ring on generators of even degree and an exterior algebra on generators of odd degree.

For the class of spaces which we consider, some important properties of minimal models were established in [2, 16]. In particular, the graded space V is described in terms of graded Lie algebras, and the dimensions of the spaces V_i are also calculated.

Theorem 1 (See [16]**).** *Let X be a simply connected closed 4-manifold with second Betti number b_2. Suppose $b_2 \geq 2$. Then*

$$\operatorname{Hom}(V_{*+1}, \mathbb{Q}) \cong \pi_*(\Omega X) = L_X = \mathbb{L}\langle x_1, \dots, x_{b_2}\rangle / \sum \varepsilon_{ij}[x_i, x_j],$$

where $\mathbb{L}$ is a free graded commutative Lie algebra, $\dim x_i = 1$, $[\cdot,\cdot]$ *is a graded commutator and (ε_{ij}) is the matrix of the comultiplication*

$$H_4(X,\mathbb{Q}) \longrightarrow H_2(X,\mathbb{Q}) \otimes H_2(X,\mathbb{Q}).$$

Theorem 2 (See [1, 2]**).** *Let X be a simply connected closed 4-manifold with second Betti number b_2. Suppose $b_2 > 2$. Then*

$$\begin{aligned}\dim V_{j+1} & \\ &= \operatorname{rk} \pi_{j+1}(X) \\ &= \frac{(-1)^j}{j} \sum_{d|j} (-1)^d \mu\left(\frac{j}{d}\right)\left(\left(\frac{2}{b_2+\sqrt{b_2^2-4}}\right)^d + \left(\frac{2}{b_2-\sqrt{b_2^2-4}}\right)^d\right),\end{aligned}$$

where μ is the Möbius function.

The following statement gives us a procedure to construct the Sullivan minimal model for the total space of the fibration $X^{S^1} \xrightarrow{\Omega X} X$ in terms of the minimal model of X itself. The Sullivan minimal model of X^{S^1} can be used to calculate $H^*(X^{S^1})$.

Theorem 3 (See [21]**).** *Let X be a simply connected space such that*

$$\dim \pi_i(X) \otimes \mathbb{Q} < \infty \quad \text{for all } i.$$

Let $(\Lambda V, d)$ be the Sullivan minimal model for X. Then $(\Lambda V \otimes \Lambda \overline{V}, d)$ is the minimal model of X^{S^1}, where $\overline{V}$ is defined as $\overline{V}_k \cong V_{k+1}$, and d is given on multiplicative generators by the formulas

$$d(v \otimes 1) = d(v) \otimes 1, \qquad d(1 \otimes \overline{v}) = -\beta(d(v)),$$

where $\beta : \Lambda V \to \Lambda V \otimes \overline{V}$ is defined on generators as $\beta(x) = \overline{x}$ and satisfies the Leibniz rule

$$\beta(a \cdot b) = (-1)^{(\deg a - 1)\deg b} b \cdot \beta(a) + (-1)^{\deg a} a \cdot \beta(b).$$

For a simply connected closed 4-manifold, the cohomology algebra and therefore the Sullivan minimal model $(\Lambda V, d)$ is defined by $\dim H^2(X) = b_2$ and the intersection form. It can be shown that for $b_2 = 0, 1, 2$ the algebra $\Lambda V \otimes \Lambda \overline{V}$ has a finite number of multiplicative generators. Therefore,