The Essential Norm of a Generalized Composition Operator Between Bloch-Type Spaces and $Q_K$ Type Spaces

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Abstract Let $\varphi$ be an analytic self-map of the unit disk $\mathbb{D}$, $H(\mathbb{D})$ the space of analytic functions on $\mathbb{D}$ and $g \in H(\mathbb{D})$. We define a linear operator as follows

$$C_{\varphi}^g f(z) = \int_0^z f'(\varphi(w))g(w) \, dw,$$

on $H(\mathbb{D})$. In this paper, estimates for the essential norm of the generalized composition operator between Bloch-type spaces and $Q_K$ type spaces are obtained.

Keywords Essential norm · Generalized composition operator · Bloch-type space · $Q_K$ type space

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1 Introduction

First, we introduce some basic notations which are used in this paper. The unit disk in the finite complex plane $\mathbb{C}$ will be denoted by $D = \{ z \in \mathbb{C} : |z| < 1 \}$. $H(D)$ will denote the space of all analytic functions on $D$, $B(D)$ will denote the subset of $H(D)$ consisting of these $f \in H(D)$ for which $|f(z)| < 1$, $dA$ will denote the Lebesgue measure on $D$, normalized so that $A(D) = 1$. For every analytic self-map $\varphi$ of the unit disk $D$ and $g \in H(D)$, Li and Stević [1,2] defined the following operator

$$C^g_\varphi f(z) = \int_0^z f'(\varphi(w)) g(w) \, dw,$$

on $H(D)$. The operator $C^g_\varphi$ is called the generalized composition operator. A fundamental problem in the study of generalized composition operators is to relate function theoretic properties of $\varphi$ and $g$ to operator theoretic properties of the restriction of $C^g_\varphi$ to various Banach spaces of analytic functions.

Recall that a bounded linear map $T$ from a Banach space $X$ into a Banach space $Y$ is called compact if it maps the closed unit ball of $X$ onto a relatively compact set in $Y$. The essential norm of $T$ is defined to be the distance to the compact operators, that is

$$\|T\|_{e,X \to Y} = \inf \{ \|T - S\|_{X \to Y} : S \text{ is compact} \}.$$

Since the set of all compact operators is a closed subset of the set of bounded operators, it follows that $\|T\|_{e,X \to Y} = 0$ if and only if $T$ is compact, estimates for $\|T\|_{e,X \to Y}$ give the conditions for $T$ to be compact.

Essential norm formulas for composition operators are known in various settings. When $C_\varphi$ acts from the Hardy space $H^2(D)$ to itself, Shapiro [3] gave a formula for $\|C_\varphi\|_e$, the essential norm of $C_\varphi$, in terms of the Nevanlinna counting function for $\varphi$. A similar formula, using a generalized Nevanlinna counting function, for the essential norm of $C_\varphi$ acting on the Bergman space $A^2(D)$ was given in [4]. In the case of the Bloch space, Montes-Rodriguez [5] gave an exact formula, namely,

$$\|C_\varphi\|_e = \lim_{r \to 1} \sup_{|\varphi(z)| > r} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} |\varphi'(z)|.$$

For the weighted composition operator from Bloch-type space $B^\alpha$ into $B^\beta (0 < \alpha < 1, 0 < \beta < \infty)$, MacCluer and Zhao [6] showed that

$$\|uC_\varphi\|_e = \lim_{r \to 1} \sup_{|\varphi(z)| > r} \frac{(1 - |z|^2)^\beta}{(1 - |\varphi(z)|^2)^\alpha} |u(z)\varphi'(z)|.$$

In [7], Stević gave upper and lower estimates for $\|DC_\varphi\|_e$ when $DC_\varphi$ maps $B^\alpha$ to $H^\infty$, where $D$ is the differentiation operator. In [8], Mikael, Makhmutov and Taskinen gave