Analytic Functions in Smirnov Classes $E^p$ with Real Boundary Values

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Abstract Smirnov domains with non-smooth boundaries do admit non-trivial functions of Smirnov class with real boundary values. We will show that the existence of functions in Smirnov classes with real boundary values is directly dependent on the boundary characteristics of a Smirnov domain.

Keywords Smirnov classes · Hardy classes · Boundary values

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Let $G$ be a bounded, simply connected domain in the complex plane with a Jordan rectifiable boundary and let $\Gamma$ be the boundary of $G$. Let $\varphi : \mathbb{D} \rightarrow G$ be the conformal mapping of the unit disk $\mathbb{D}$ onto $G$. Recall (cf. [1] Ch. 10.)

Definition 1 An analytic function $f(z)$ in $G$ is said to belong to the class $E^p(G)$ for $p$ such that $0 < p \leq \infty$ if there exists a sequence of rectifiable curves $\{\Gamma_i\}$ in $G$ converging to $\Gamma$ such that

$$\int_{\Gamma_i} |f(z)|^p |dz| \leq \text{const}. < \infty.$$
Definition 2 G is called a Smirnov domain if \( \varphi(w) \) is an outer function.

It is well known that for \( p \geq 1 \) functions in \( E^p \)-classes can be represented as Cauchy integrals of their boundary values. Their counterparts, functions in Hardy classes \( H^p(G) \), \( p \geq 1 \), are conformally invariant and representable as Poisson integrals of their boundary values (cf. [2], Ch. 4, [6]). So if \( f(z) \in H^p(G) \), \( p \geq 1 \) is real a.e. on \( \Gamma \), then \( f(z) \) must be real valued in \( G \) and hence is a constant. As is well known, (cf. [1] Ch. 10), if \( \Gamma \) is smooth, there exist constants \( c_1 \) and \( c_2 \) such that \( 0 < c_1 < |\varphi'(w)| < c_2 < \infty \) in \( \mathbb{D} \) implying that \( H^p(G) = E^p(G) \) as sets for all \( p \) and hence, the observation holds for Smirnov classes as well (e.g., \( G = \mathbb{D} \)).

An elegant theorem of Newman and Neuwirth [3] states that if \( f(w) \in H^p(\mathbb{D}) \) for \( p \geq 1/2 \) and \( f(w) \geq 0 \) a.e. on \( \mathbb{T} = \partial \mathbb{D} \), then \( f(w) \equiv \text{const} \). The Koebe function \( f(w) = \frac{w}{(1+w)^2} \) furnishes an example for all \( p < 1/2 \). It has been proven in [4] that for non-Smirnov domains for \( 0 < p < \infty \) there exists a non-constant \( f(z) \in E^p(G) \) such that \( 0 \leq f(z) \leq 1 \) a.e. on \( \Gamma \). It is also noted in [4] that if \( G \) is a Smirnov domain, \( f(z) \in E^1(G) \), and \( f(z) \) is real and bounded on \( \Gamma \) then \( f(z) \equiv \text{const} \). However, when the condition of boundedness is removed, an example by S. Ya. Khavinson [4] reveals that for a certain “cardioid type” Smirnov domains the function \( f(z) = F(\varphi^{-1}(z)) \) where \( F(w) = \frac{i}{1+w}+\frac{w}{1-w} \) and \( \varphi(w) = (1-w)^2 \) furnishes a non-trivial \( E^1 \) function with real boundary values. In fact, the example works for all \( 0 < p < 2 \). S. Ya. Khavinson’s example illustrates how the existence of \( E^p \) functions with real boundary values is dependent upon the boundary properties of the domain. This paper will delineate the types of Smirnov domains with non-smooth boundaries that admit \( E^p \) functions with real boundary values.

To begin, let us consider the simple case of a domain bounded by a curve that is analytic except at one point at which there is a corner of angle \( \alpha \). It turns out that the existence of \( E^p \) functions with real boundary values is directly dependent upon the angle \( \alpha \). The following theorem which can be found in [1, Sec. 10.1] will be useful and we state it here for the reader’s convenience.

Theorem A (M. Keldysh, M. Lavrentiev)

\[
f(z) \in E^p(G) \Leftrightarrow F(w) = f(\varphi(w))[\varphi'(w)]^{1/p} \in H^p(\mathbb{D})
\]

for some conformal mapping \( \varphi(w) \) of the unit disk onto \( G \).

We shall start with a simple observation.

Theorem 1 Let \( G \) be a domain bounded by a curve that is real analytic except at the point \( z_0 \) where it has a corner with interior angle \( \alpha \). If \( 0 \leq \alpha \leq \pi \), then for all \( p \geq 1 \) every \( f(z) \in E^p(G) \) with real boundary values is a constant.

Note. The angle \( \alpha = 0 \) corresponds to an outward cusp. The angle \( \alpha = \pi \) corresponds to \( \Gamma \) being smooth and was discussed above.

Proof Let \( 0 < \alpha < \pi \) and suppose that there exists an \( f(z) \in E^1(G) \) with real boundary values a.e. Without loss of generality we may assume that \( \varphi(1) = z_0 \). According to [5, Ch. 3, Sec. 4]