Space Saving Calculation of Symbolic Resultants

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Abstract. We describe an approach to the computation of symbolic resultants in which factors are removed during the course of the calculation, so reducing the stack size required for intermediate expressions and the storage space needed. We apply the technique to three well-established methods for calculating resultants. We demonstrate the advantages of our approach when the resultants are large and show that some otherwise intractable problems can be resolved. In certain cases a significant reduction in the cpu time required to calculate the resultant is also evident.

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1. Introduction

Our interest in the calculation of symbolic resultants arose from our research into some of the properties of systems of nonlinear differential equations (see for example [9–12]). In order to provide a context for our discussion we first give a brief description of the mathematical background. We consider differential systems of the form

\[ \dot{x} = \lambda x + y + p(x, y), \quad \dot{y} = -x + \lambda y + q(x, y), \quad (1) \]

where \( p \) and \( q \) are polynomials without linear or constant terms. The corresponding complex form of system (1) is

\[ i \dot{z} = (1 + i\lambda)z + \sum_{i+j=2} A_{ij} z^i \bar{z}^j, \quad (2) \]

where \( z = x + iy \), \( A_{ij} \in \mathbb{C} \). When \( \lambda = 0 \) the origin is said to be a fine focus. Our objective is to establish the conditions under which the origin is a centre and to determine the maximum number of limit cycles that can be bifurcated from the origin for systems of the form (1) or (2) under perturbation of the coefficients. All orbits in the neighbourhood of the origin when it is a centre are closed; in contrast a limit cycle is an isolated closed orbit.
We separate the calculation of the conditions for the origin to be a centre into two parts: necessity and sufficiency. It is in the calculation of the necessary conditions that most of the large resultant calculations arise. These conditions are obtained by calculating the focal values, which are polynomials in the coefficients in $p$ and $q$, (or in the $A_{ij}$) and are defined as follows. There is a function $V$, analytic in a neighbourhood of the origin, such that the rate of change along orbits, $\dot{V} = \eta_2 r^2 + \eta_4 r^4 + \cdots$, where $r^2 = x^2 + y^2$ and $\eta_2 = \lambda$. The $\eta_{2k}$ are the focal values and the number of terms in each $\eta_{2k}$ grows rapidly as $k$ increases. In examples where $\eta_4$ has only four terms it is not unusual for $\eta_{12}$ to have over a thousand terms, and $\eta_{14}$ over three thousand.

The origin is a centre if, and only if, all the focal values are zero. By the Hilbert basis theorem, the ideal they generate has a finite basis so there is $M$ such that if $\eta_{2j} = 0$, for $j \leq M$, then $\eta_{2j} = 0$ for all $j$. The value of $M$ is not known a priori. The origin is a fine focus of order $k$ if $\eta_{2m} = 0$, for $m \leq k$ but $\eta_{2k+2} \neq 0$.

Our approach is to calculate the first few focal values for a given system and to make substitutions from each one into the other focal values. We have $\eta_2 = \lambda = 0$, necessarily. We make a substitution from $\eta_4 = 0$, for one of the variables, into subsequent calculated focal values. Then we make substitutions from $\eta_6 = 0$, $\eta_8 = 0$ and so on. We thus obtain expressions for each of the eliminated variables in terms of the remaining variables – information that is required for the bifurcation of the limit cycles. After each substitution we remove common factors from the remaining calculated focal values, these being candidates for the conditions under which the origin is a centre. We continue until we can show that if the remaining factor of focal value $\eta_{2k}$ is zero then focal value $\eta_{2k+2}$ is necessarily non-zero. Then the maximum order of the origin as a fine focus is $k$.

The sufficiency of the candidate centre conditions is proved independently using a range of different techniques.

We have considered looking for a Gröbner basis for the set of focal values; there are three main drawbacks to this approach. First, we do not know a priori the value of $M$ for a given differential system. Secondly, the Gröbner basis does not readily give us the information we require in order to bifurcate the limit cycles. Finally, obtaining the Gröbner basis is non-trivial for many systems.

In the systems of interest to us the focal values usually involve at least seven variables. As each variable is eliminated the remaining focal values grow; they contain more terms, the variables occur to higher degrees and the integer coefficients become larger. At each stage of the elimination process an attempt is made to simplify the focal values by factorising them, each such factor is then considered individually. However we inevitably reach a point where the variable we wish to eliminate does not occur linearly in any of the focal values (or factors of the focal values) and we must employ polynomial remainder sequences, as in [10,12] or use resultant calculations, see for example [12], to eliminate that variable. Often successive resultant calculations are required; the performance of such calculations is sensitive to the order in which the variables are eliminated.