On Partitional Labelings of Graphs

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Abstract The notion of partitional graphs, a subclass of sequential graphs, is introduced, and the cartesian product of a partitional graph and $K_2$ is shown to be partitional. Every sequential graph is harmonious and felicitous. The partitional property of some bipartite graphs including the $n$-dimensional cube $Q_n$ is studied, and thus this paper extends what was known about the sequentialness, harmoniousness and felicitousness of such graphs.

Keywords Partitional labeling · Sequential labeling · Harmonious labeling · Felicitous labeling · $n$-Cube · Graph labeling

Mathematics Subject Classification (2000) 05C78

1 Introduction

Throughout this paper, we consider a graph $G$ to be finite and simple, that is, there are no loops and multiple edges. Let $V(G)$ and $E(G)$ denote the vertex and edge sets of $G$, respectively. For any undefined graph theory terminology and notation, the authors refer the reader to Chartrand and Lesniak [1]. For the sake of brevity, we will denote the set of integers $\{m, m+1, \ldots, n\}$ by simply writing $[m, n]$.

Harmonious graphs naturally arose in the study by Graham and Sloane [7] of modular versions of additive bases problems stemming from error-correcting codes. They defined a graph $G$ of order $p$ and size $q$ with $q \geq p$ to be harmonious if there exists an injective function $f : V(G) \rightarrow \mathbb{Z}_q$ such that when each $uv \in E(G)$ is labeled $f(u) + f(v) \pmod q$, the resulting edge labels are distinct. Such a function is called a harmonious labeling. If $G$ is a tree so that $q = p - 1$, exactly two vertices are labeled the same; otherwise, the definition is the same.
In [6], Grace defined a sequential labeling of a graph $G$ of size $q$ as an injective function $f : V(G) \rightarrow [0, q-1]$ (with the label $q$ allowed if $G$ is a tree) such that when each $uv \in E(G)$ is labeled $f(u) + f(v)$, the resulting edge labels are $[m, m+q-1]$ for some positive integer $m$. A graph is called sequential if it admits a sequential labeling. Every sequential labeling induces a harmonious labeling; however, it is an open question whether or not every harmonious graph admits a sequential labeling.

Harmonious and sequential labelings have been the object of study for many papers. For recent contributions to these subjects and other types of labelings, the authors refer the reader to an excellent survey paper by Gallian [5].

To present our results, we define the cartesian product of two graphs. The cartesian product $G \cong G_1 \times G_2$ has $V(G) = V(G_1) \times V(G_2)$, and two vertices $(u_1, u_2)$ and $(v_1, v_2)$ of $G$ are adjacent if and only if either $u_1 = v_1$ and $u_2v_2 \in E(G_2)$
or$u_2 = v_2$ and $u_1v_1 \in E(G_1)$.

In particular, the $n$-dimensional cube $Q_n$ can be defined inductively as $Q_0 \cong K_1$ and $Q_n \cong Q_{n-1} \times K_2$ for any positive integer $n$.

In Sect. 2, we show that $Q_n$ is sequential for every integer $n \geq 4$. In Sect. 3, we introduce the notion of a partitional labeling, which is a sequential labeling with additional properties and, generalizing the argument in Sect. 2, prove that the cartesian product of a partitional graph and $K_2$ is partitional (see Theorem 3.1). It is clear that every partitional labeling is sequential and harmonious. It is also true that every partitional graph is ‘felicitous’ (see [5] for the definition of a felicitous labeling). Thus, if a graph $G$ is shown to be partitional, then by virtue of Theorem 3.1, it follows that for each nonnegative integer $n$, $G \times Q_n$ is sequential, harmonious and felicitous (see Theorem 3.4, and Corollaries 3.7 and 3.8 in Sect. 3).

## 2 Result on the $n$-Dimensional Cube

The $n$-dimensional cube $Q_n$ (also called hypercube) serves as useful models for a broad range of applications such as circuit design, communication network addressing, parallel computation and computer architecture. This motivates us to examine in this section the sequential labeling of $Q_n$.

The following proof is inspired by the work that Kotzig [8] carried out when showing that there exists an $\alpha$-valuation of $Q_n$ for every positive integer $n$. Utilizing the construction in his proof, Figueroa-Centeno and Ichishima [3] have recently verified that every $n$-dimensional cube $Q_n$ is felicitous.

**Theorem 2.1** Let $n$ be an integer with $n \geq 2$. Then the $n$-dimensional cube $Q_n$ is sequential if and only if $n \neq 2, 3$.

**Proof** The $n$-dimensional cube $Q_n$ has already been shown to be not harmonious by Graham and Sloane [7] when $n = 2, 3$. Thus, they are not sequential either.

For the converse, assume that $n$ is an integer with $n \geq 4$, and proceed by induction on $n$. First, let $Q_2$ be the graph with $V(Q_2) = \{v_1, v_2, v_3, v_4\}$ and $E(Q_2) = \{v_1v_3, v_1v_4, v_2v_3, v_2v_4\}$. For every integer $n \geq 2$, construct the graph $Q_{n+1}$ by using the decomposition

$Q_{n+1} \cong A_{n+1} \oplus B_{n+1} \oplus C_{n+1}$

with

$V(A_{n+1}) = \{v_i | i \in \left[1, 2^{n-1}\right]\} \cup \{v_i | i \in \left[2^n + 1, 2^{n-1} + 2^n + 2^{n-1}\right]\}$,

$V(B_{n+1}) = \{v_i | i \in \left[1, 2^{n+1}\right]\}$,

$V(C_{n+1}) = \{v_i | i \in \left[2^{n-1} + 1, 2^n\right]\} \cup \{v_i | i \in \left[2^n + 2^{n-1} + 1, 2^{n+1}\right]\}$,