The Geometry of Standard Deontic Logic

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Abstract. Whereas geometrical oppositions (logical squares and hexagons) have been so far investigated in many fields of modal logic (both abstract and applied), the oppositional geometrical side of “deontic logic” (the logic of “obligatory”, “forbidden”, “permitted”, ...) has rather been neglected. Besides the classical “deontic square” (the deontic counterpart of Aristotle’s “logical square”), some interesting attempts have nevertheless been made to deepen the geometrical investigation of the deontic oppositions: Kalinowski (La logique des normes, PUF, Paris, 1972) has proposed a “deontic hexagon” as being the geometrical representation of standard deontic logic, whereas Joerden (jointly with Hruschka, in Archiv für Rechtsund Sozialphilosophie 73:1, 1987), McNamara (Mind 105:419, 1996) and Wessels (Die gute Samariterin. Zur Struktur der Supererogation, Walter de Gruyter, Berlin, 2002) have proposed some new “deontic polygons” for dealing with conservative extensions of standard deontic logic internalising the concept of “supererogation”. Since 2004 a new formal science of the geometrical oppositions inside logic has appeared, that is “n-opposition theory”, or “NOT”, which relies on the notion of “logical bi-simplex of dimension m” (m = n – 1). This theory has received a complete mathematical foundation in 2008, and since then several extensions. In this paper, by using it, we show that in standard deontic logic there are in fact many more oppositional deontic figures than Kalinowski’s unique “hexagon of norms” (more ones, and more complex ones, geometrically speaking: “deontic squares”, “deontic hexagons”, “deontic cubes”, …, “deontic tetraicosahedra”, …): the real geometry of the oppositions between deontic modalities is composed by the aforementioned structures (squares, hexagons, cubes, …, tetraicosahedra and hyper-tetraicosahedra), whose complete mathematical closure happens in fact to be a “deontic 5-dimensional hyper-tetraicosahedron” (an oppositional very regular solid).

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Figure 1. The main features of the logical square of opposition

Figure 2. Classic and modern quantificational and modal squares

1. The “Logical Square” Inside Deontic Logic: the “Deontic Square”

The logical square is an object synthesising many fundamental properties of formal logic. Its two upper terms are universal, its two lower terms are particular, the two terms on the left side are positive, the two at the right side are negative. Its four conventional constitutive colors (blue, red, green, black) represent the four possible Aristotelian ways of being “opposed” (respectively: contrariety, contradiction, subcontrariety and subalternation)\(^1\) (cf. Fig. 1).

This abstract geometric structure (of which we will speak in more detail in one of the next paragraphs) admits many interpretations according to the “objects” (logical or not) used to decorate its empty places (the empty places, in the figure, are the bigger dots - the 4 corners). One interpretation seems to be primordial, that is the “quantificational interpretation”, the one displaying over the square the values “all” (\(\forall\)), “some” (\(\exists\)), “none” (\(\forall \neg\)) and “some not” (\(\exists \neg\)). Then a second very important interpretation is the “modal” one, or, more precisely (for there are different families of modal logics), the “alethic” (or “ontic”) one, the one displaying over the square the classical modal values “necessary”, “possible”, “impossible” and “possible that not” (cf. Fig. 2).

These two interpretations were given by Aristotle himself.\(^2\) Nowadays we know, thanks to Kripke’s et alii’s (Hintikka, Kanger, …) “possible world semantics” (which we will recall briefly later), that this fact - the parallel presence of a quantificational and of a modal reading - is so to say normal: for modal logic is generated by introducing suited restrictions to “quantification

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\(^1\)We discuss extensively in [25] and [29] the question of knowing whether all the four Aristotelian ingredient relations of the logical square (i.e. contrariety, contradiction, subcontrariety and subalternation) are opposition relations, for it is often objected that subcontrariety and subalternation are not oppositions. We argue strongly that they all are.

\(^2\)The graphical square is only attested by later Latin thinkers: Apuleius, Boetius. We actually do not know if Aristotle himself knew it (for sure he knew—he created, mainly in the Peri hermeneias—the underlying logic, cf. [31] and [38]). We will call it though, following the traditional use, “Aristotle’s square”.