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A Hexagonal Framework of the Field $\mathbb{F}_4$ and the Associated Borromean Logic

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Friendly dedicated to Jacques Riguet, on the occasion of his 90th birthday

Abstract. The hexagonal structure for 'the geometry of logical opposition', as coming from Aristoteles–Apuleius square and Sesmat–Blanché hexagon, is presented here in connection with, on the one hand, geometrical ideas on duality on triangles (construction of 'companion'), and on the other hand, constructions of tripartitions, emphasizing that these are exactly cases of borromean objects. Then a new case of a logical interest introduced here is the double magic tripartition determining the semiring $\mathcal{B}_3$ and this is a borromean object again, in the heart of the semiring $\mathrm{Mat}_3(\mathbb{B}_{\mathrm{Alg}})$. With this example we understand better in which sense the borromean object is a deepening of the hexagon, in a logical vein. Then, and this is our main objective here, the Post-Mal'cev full iterative algebra $\mathbb{P}_4 = \mathbb{P}(\mathbb{F}_4)$ of functions of all arities on $\mathbb{F}_4$, is proved to be a borromean object, generated by three copies of $\mathbb{P}_2$ in it. This fact is induced by a hexagonal structure of the field $\mathbb{F}_4$. This hexagonal structure is seen as precisely a geometrical addition to standard boolean logic, exhibiting $\mathbb{F}_4$ as a 'boolean manifold'. This structure allows to analyze also $\mathbb{P}_4$ as generated by adding to a boolean set of logical functions a very special modality, namely the Frobenius squaring map in $\mathbb{F}_4$. It is related to the splitting of paradoxes, to modified logic, to specular logic. It is a setting for a theory of paradoxical sentences, seen as computations of movements on the bi-hexagonal link among the 12 classical logics on a set of 4 values.

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1. Logics Coming from Borromean Objects

The idea of a hexagonal setting in logic of opposition could be deepened, in a 'functional' direction (i.e. in order to show and analyze algebras of functions) by the notion of a borromean object, recalled here.

Definition. A standard (or resp. a reduced) *borromean object* in a category $\mathcal{C}$ with null morphisms (and a terminal and initial object), cokernels and finite sums (finite coproducts), is (definition 1s-1r, [11, p. 145]) an object B equipped with three objects R, S, T in $\mathcal{C}$ and an epimorphic family of monomorphisms in $\mathcal{C}$,

$$r : R \to B, \quad s : S \to B, \quad t : T \to B$$

such that $B/r \simeq S + T, B/s \simeq T + R, B/t \simeq R + S$ (or resp. such that $B/r \simeq 1, B/s \simeq 1, B/t \simeq 1$).

So the standard case is pictured in $\mathcal{C}$ as β_B:

$$\begin{tikzcd}
 & R \arrow[rr] \arrow[dl] \arrow[dr, tail, "r"] & & R+S & \\
T+R & & B \arrow[ll, two heads] \arrow[ur, two heads] \arrow[dr, two heads] & & S \arrow[ul] \arrow[ll, tail, "s"] \arrow[dl] \\
 & T \arrow[ul] \arrow[ur, tail, "t"] \arrow[rr] & & S+T &
\end{tikzcd}$$

In the category of finite boolean algebras this notion is equivalent to a tripartition of a set (Proposition 3.4). We have also the case of pointed 3-partitions (Proposition 3.5). In the category of semi-rings, a logically meaningful example is the semi-ring $\mathcal{B}_3$ (Proposition 4.1). In the category of groups we get the examples of the fundamental group of the complement of a *borromean link*, of the groups $\mathcal{S}(3)$, $\mathbb{Z}/7\mathbb{Z}$, and—more sophisticated—the Klein's group G_{168} of the Klein's quartic (cf. [11]).

Of course the diagram of a borromean object has basically a hexagonal framework, but its logico-geometrical scheme β is deeper than the simple picture of a hexagon. Especially the 'opposition' in a borromean diagram does not construct the opposite as a complement, but as a quotient or a dual. Furthermore the central object B is not determined by its components R, S, T, and in fact has to be thought as a new datum, the datum of an original link between the components.

It happens that if the objects of $\mathcal{C}$ have a 'logical meaning' (e.g. if they are boolean algebras), then a given borromean object B in $\mathcal{C}$ generates a logico-geometrical borromean system of such 'logical meanings'—denoted by $\beta[B]$—and such a mixture is a kind of cross product of the pure abstract logico-geometrical scheme β with the inner logical content of each object in the diagram of B. We will show in detail such a structure in the case of the tripartition of a set,