Direct Proofs of Lindenbaum Conditionals

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Abstract. We discuss the problem raised by Miller (Log Univers 1:183–199, 2007) to re-prove the well-known equivalences of some Lindenbaum theorems for deductive systems (each equivalent to the Axiom of Choice) without an application of the Axiom of Choice. We present five special constructions of deductive systems, each of them providing some partial solutions to the mathematical problem. We conclude with a short discussion of the underlying philosophical problem of deciding, whether a given proof satisfies our demand that the Axiom of Choice is not applied.

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1. Introduction

It is a statement of facts, when, for example, Arana writes that many “mathematicians and philosophers of mathematics think that it is somehow valuable for a proof to be ‘pure’, that is, not to use notions extraneous to what is being proved.”\(^1\) With analogous argumentation, Felgner justifies the demand of mathematicians outside set theory for exploring such theorems as Zorn’s Lemma, which are equivalent to the Axiom of Choice (AC) and to the Well-Ordering Theorem, but do not use the notion of ordinals:

Outside set theory, [..] one legitimately feels uneasy that one should apply in the proofs the theory of ordinal numbers, while in

\(^1\) Cf. [1, p. 189].

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the formulations of the theorems which are to be proved no ordinal
numbers occur at all.\footnote{Cf. \cite[§13, p. 134]{4}, translation by the author.}

Miller \cite{11} stays in this tradition, when he discusses a result given by
Dzik \cite{3} that the generalised version of Lindenbaum’s Theorem (LB), in the
form that every deductive system \textit{allows saturation}, implies AC.\footnote{It is well-known that conversely AC implies LB. Therefore, both theorems are equivalent. For a proof see, for example, Miller \cite[§2, theorem 2.5]{11}.}

Analysing Dzik’s proof, Miller observes that LB is only applied to a very special class of deductive systems and, therefore, that a restricted version of LB is sufficient to prove AC. He defines this restricted Lindenbaum theorem together with some intermediate versions and concludes that all of them are equivalent.

While those equivalences are still proved in the non-trivial direction via the Axiom of Choice, which is a theorem external to the theory of deductive systems, Miller raises the question, whether there are pure proofs of them without an application of AC.\footnote{We mention that Miller provides a reason of re-proving already proved theorems. This phenomenon is considered by Kahle \cite{8} as an important aspect of the philosophical notion of proofs.}

He succeeds to establish two non-trivial conditionals of those in question by providing two special constructions for deductive systems; the proofs are elementary and direct. Intuitively, it is clear that Miller satisfies thereby his own demands.

Concentrating on the mathematical problem, we briefly recall the set theoretical theory of deductive systems as given by Tarski \cite{13} and introduce the mathematical concepts needed in this paper. In the succeeding section we discuss Miller’s observations and his demands in some detail.

In the main part of this paper, we follow Miller’s approach and provide five special constructions, each allowing to prove some further, non-trivial conditionals in question.

We conclude with a brief discussion of some philosophical aspects of the mathematical problem; in particular, we illustrate the underlying philosophical problem of deciding, whether a given proof satisfies our demands, and we sketch how to find a good criterion in principle.

\textbf{Notation} (Set theory). We assume in this paper (almost informally) set theory ZF formalised in a first order language.\footnote{A formalisation of ZF is found in \cite{7}.} We use the following set theoretical notation.

1. \textit{set}: The variable $S$ denotes, if not specified otherwise, an arbitrary set of arbitrary cardinality representing an unspecified formal language.
2. \textit{subset}: We use the abbreviation $X \subseteq \omega Y$ to mean that a set $X$ is a subset of a set $Y$ and additionally that the set $X$ is finite. Similarly, we write $X \subseteq_n Y$ to indicate $|X| \leq n$ for a given natural number $n \in \mathbb{N}$.
3. \textit{power set}: With $p(S) = \{X; X \subseteq S\}$ we denote the power set of $S$, with $p_{\omega}(S) = \{X \subseteq S; \text{ X is finite}\}$ the set of all finite subsets of $S$.