Study on High Order Perturbation-based Nonlinear Stochastic Finite Element Method for Dynamic Problems

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Abstract: Several algorithms were proposed relating to the development of a framework of the perturbation-based stochastic finite element method (PSFEM) for large variation nonlinear dynamic problems. For this purpose, algorithms and a framework related to SFEM based on the stochastic virtual work principle were studied. To prove the validity and practicality of the algorithms and framework, numerical examples for nonlinear dynamic problems with large variations were calculated and compared with the Monte-Carlo Simulation method. This comparison shows that the proposed approaches are accurate and effective for the nonlinear dynamic analysis of structures with random parameters.

Keywords: high-order; stochastic variational principle; nonlinear SFEM; perturbation technique

1 Introduction

The analysis of structural systems with uncertain properties modeled by random fields has been the subject of extensive research in the past two decades. Over these years, the majority of the research work has focused on developing various stochastic finite element method (SFEM) for the numerical solution of the stochastic partial differential equations involved in such problem, i.e. stochastic spectral approaches (Ghanem and Spanos, 2003), a variety of Monte-Carlo simulation techniques (Marcin Kaminski, 2007) (Liu et al., 1986) as well as many numerical realizations of the perturbation technique (Kleiber and Hien, 1992).

Since nonlinear analysis of structures with stochastic parameters is of considerable importance as many of the buildings, offshore structures, ships, etc., some dynamic excitations that are excited by nature’s actions which exhibit randomly fluctuating character cannot be treated as deterministic systems. It is essential to consider nonlinearity arising from geometrical and/or material properties in random structure dynamics. Liu and Kiureghian (1988), Haldar and Zhou (1992) all studied the solutions of two-dimensional nonlinear SFEM by the method of partial differentiation. Hisada and Noguchi (1989) proposed perturbation SFEM to solve material’s nonlinear problems. Liu et al. (1989) applied the first order perturbation technique to solve the nonlinear dynamic response of random structure. Zhao and Chen (2000) used minimum potential energy principle and perturbation technique to solve nonlinear dynamic problem. Ioannis and Zhan (2006) addressed the perturbation-based stochastic finite element analysis to study deformation processes of inelastic solids.

The basic idea of the stochastic perturbation approach is to expand all the input variables and all the state functions of the given problem via Taylor series about their spatial expectations using some small parameter \( \varepsilon > 0 \). As introduced above, the second-order perturbation approach has the well-known limitations on the input coefficients of variation: the variances of random variables should be small when compared with their expected values for accuracy. This limitation restricts the analysis to some limits of random fields, so it is necessary to study large variation of random variables in stochastic problems, especially in nonlinear dynamic problems. However, at present, very few researches have been carried out about these. Kaminski M. used minimum potential energy principle to study generalized perturbation-based SFEM to model 1-D linear elastostatics problem with a single random variable in elastostatics in 2007. So in this thesis, an extended approach of generalized perturbation-based stochastic variational method for nonlinear dynamic problems with single random variable is proposed and studied.

2 Theoretical backgrounds

In update Lagrangian descriptions the last known configuration was adopted as the reference state, the region taken up by the body at this instant will be denoted by \( ^i\Omega \), and the primary equations describing the incremental problem may be presented in the time interval \( [t, t + \Delta t] \), the first and second Piola-Kirchhoff stress tensors \( \Delta S^{(i)}_{ij} \) and \( \Delta S^{(d)}_{ij} \) based on the current configuration are related by the equation (Kleiber and Hien, 1992):
\[ \Delta S^{(i)} = \Delta S^{(0)} + \Delta \mu \cdot \tau_{ij} \]  
(1)

where, \( \tau_{ij} \) is Cauchy stress tensor in the configuration at time \( t \), so the weighted residual formulation is

\[ \int_\Omega \left( \Delta S^{(i)} + \rho \Delta f,_{i} - \rho \Delta \mu,_{ij} \right) \delta (\Delta, u) d\Omega + \int_{\delta \Omega} (\Delta, \tilde{j} - \Delta S^{(i)}(n)) \delta (\Delta, u) dS = 0 \]  
(2)

The integral by parts on the first part of left hand side is as follows:

\[ \int_\Omega \Delta S^{(i)} d\Omega = \int_\Omega \Delta S^{(i)}(n) dS - \int_\Omega \Delta S^{(i)}(n) \delta (\Delta, u) d\Omega \]  
(3)

By employing Gauss-Ostrogradski theorem,

\[ \int_\Omega A \delta d\Omega = \int_{\delta \Omega} A_{n} dS \]

Eq. (3) becomes

\[ \int_\Omega \Delta \nabla^{(i)}(\Delta, u) d\Omega = \int_{\delta \Omega} \nabla^{(i)}(\Delta, u) n dS - \int_\Omega \Delta \nabla^{(i)}(\Delta, u) d\Omega \]  
(4)

where, \( S = S_{u} \cup S_{\mu} \), from the incremental primary equations we can deduce:

\[ \delta (\Delta, u) = \delta (\Delta, u) = 0 \]  

(5)

\[ \delta (\Delta, u) = \delta \left( \frac{1}{2} \delta (\Delta, u) + \delta (\Delta, u) \right) = \delta (\Delta, u) \]  

(6)

Substitute Eqs. (5) and (6) into Eq. (4), we have

\[ \int_\Omega \Delta \nabla^{(i)}(\Delta, u) d\Omega + \int_{\delta \Omega} \nabla^{(i)}(\Delta, u) n dS - \int_\Omega \Delta \nabla^{(i)}(\Delta, u) d\Omega = 0 \]  

(7)

By substituting Eq. (7) into Eq. (2), the weighted residual formulation becomes

\[ \int_\Omega \Delta \nabla^{(i)}(\Delta, u) d\Omega + \int_{\delta \Omega} \nabla^{(i)}(\Delta, u) n dS = 0 \]  

(8)

By expressing the first Piola-Kirchhoff stress tensor in terms of the second Piola-Kirchhoff stress tensor by Eq. (1) we have

\[ \int_\Omega \rho \Delta A,_{ij} \delta (\Delta, u) d\Omega + \int_\Omega \tau_{ij} \Delta A,_{ij} \delta (\Delta, u) d\Omega + \int_\Omega \rho \Delta A,_{ij} \delta (\Delta, u) d\Omega + \int_\Omega \Delta A,_{ij} \delta (\Delta, u) d\Omega = \int_\Omega \rho \Delta A,_{ij} \delta (\Delta, u) d\Omega + \int_\Omega \Delta A,_{ij} \delta (\Delta, u) d\Omega - \int_{\delta \Omega} \tau_{ij} \Delta A,_{ij} \delta (\Delta, u) dS \]  

(9)

where, \( \rho \Delta A,_{ij} \) is tangent stress-strain tensor at time \( t \).

Furthermore, let’s assume the damping effect is the nature of the body and is proportional to the velocities of the body particles, so Eq. (9) becomes

\[ \int_\Omega \rho \Delta A,_{ij} \delta (\Delta, u) d\Omega + \int_\Omega \rho \Delta A,_{ij} \delta (\Delta, u) d\Omega + \int_\Omega \rho \Delta A,_{ij} \delta (\Delta, u) d\Omega + \int_\Omega \Delta A,_{ij} \delta (\Delta, u) d\Omega - \int_{\delta \Omega} \tau_{ij} \Delta A,_{ij} \delta (\Delta, u) d\Omega \]  

(10)

where, the constant \( \alpha \) is a proportional damping factor.

Now let’s consider any random state function \( f(b) \), where \( b \) is a single random quantity, and its expansion via Taylor series up to \( n \)-th order is

\[ f(b) = f(b^{0}) + \frac{\partial f}{\partial b} \Delta b + \frac{1}{2} \frac{\partial^{2} f}{\partial b^{2}} \Delta b^{2} + \cdots + \frac{1}{n!} \frac{\partial^{n} f}{\partial b^{n}} \Delta b^{n} \]  

(11)

Its expectation is

\[ E[f(b)] = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^{n} f}{\partial b^{n}} \Delta b^{n} \]  

(12)

From the numerical point of view, the expansion is carried out for the summation over the finite numbers of components (Kaminski, 2007).

\[ E[f(b)] = f^{0} + \sum_{n=2}^{\infty} \frac{1}{n!} \frac{\partial^{n} f}{\partial b^{n}} \Delta b^{n} \]  

(13)

It can be done for the expected values and the variances by introducing the following statistical error measures (Oden et al., 2005):

\[ \forall \delta_{i} \in \mathbb{R}^{+} \quad \exists N_{i} \in \mathbb{N} \quad \left| E[f_{N_{i}}(b)] - E[\tilde{f}(b)] \right| \leq \delta_{i} \]  

\[ \forall \delta_{i} \in \mathbb{R}^{+} \quad \exists N_{i} \in \mathbb{N} \quad \left| \text{Var}[f_{N_{i}}(b)] - \text{Var}[\tilde{f}(b)] \right| \leq \delta_{i} \]  

The real positive numbers \( \delta_{i} \) and \( \delta_{s} \) denote the admissible errors for determination of the expectations and variances. Natural quantities \( N_{i} \) and \( N_{s} \) correspond to the orders of perturbation which result in a desired accuracy; the maximum of these two numbers fulfills satisfactory accuracy conditions. Finally, it yields

\[ E[	ilde{f}(b)] = \lim_{L \rightarrow \infty} \frac{1}{L} \sum_{i=1}^{L} f_{i}(b) \]  

as well as

\[ \text{Var}[	ilde{f}(b)] = \lim_{L \rightarrow \infty} \frac{1}{L} \sum_{i=1}^{L} [f_{i}(b) - E[	ilde{f}(b)]]^{2} \]  

where \( L \) is the total number of random trials in statistical verification of the estimators for the random function \( f(b) \). As it shows, sufficiently accurate modeling of the moments by the perturbation technique needs initial Monte-Carlo simulation computations to determine the necessary order of the perturbation for a given problem.

If higher order terms are necessary, then the following extension can be proposed:

\[ E[f(b); b] = f^{0}(b^{0}) + \frac{1}{2} \frac{\partial^{2} f}{\partial b^{2}}(\Delta b)^{2} + \frac{1}{4} \frac{\partial^{4} f}{\partial b^{4}}(\Delta b)^{4} + \frac{1}{6} \frac{\partial^{6} f}{\partial b^{6}}(\Delta b)^{6} + \cdots \]  

where all terms with the odd orders are equal to 0 for the Gaussian random deviates and those higher than the 6th order terms are neglected. According to such an extension of the random output, any desired efficiency of the expected