Analysis of onset of buoyancy-driven convection in a fluid layer saturated in anisotropic porous media by the relaxed energy method

Min Chan Kim†

Department of Chemical Engineering, Jeju National University, Jeju 690-756, Korea

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Abstract—A theoretical analysis of buoyancy-driven instability under transient basic fields is conducted in an initially quiescent, fluid-saturated, horizontal porous layer. Darcy’s law is used to explain characteristics of fluid motion, and the anisotropy of permeability is considered. Under the Boussinesq approximation, the energy stability equations are derived following the energy formulation. The stability equations are analyzed numerically under the relaxed energy stability concept. For the various anisotropic ratios, the critical times are predicted as a function of the Darcy-Rayleigh number, and the critical Darcy-Rayleigh number is also obtained. The present predictions are compared with existing theoretical ones.

Key words: Buoyancy-driven Convection, Anisotropic Porous Medium, Energy Method

INTRODUCTION

Buoyancy-driven phenomena in porous media play an important role in a wide variety of engineering applications, such as geothermal reservoirs, agricultural product storage system, packed-bed catalytic reactors and pollutant transport under the ground. One of the attractive convection phenomena in porous media is the naturally enhanced carbon dioxide (CO$_2$) dissolution into the saline water confined within the geologically stable formations [1-10]. In this particular case the heavier CO$_2$ saturated water will flow downward and will be replaced by water with lesser CO$_2$ content. Although the density increase of CO$_2$ is only around 1% at subsurface conditions, on long time scales this can lead to a convective mixing process, which significantly accelerates the dissolution of CO$_2$, and thus improves the containment.

The onset of buoyancy-driven convective instabilities in porous media was first analyzed by Horton and Rogers [11] and independently by Lapwood [12]. They examined thermally driven convection and used the methods developed for convection in a homogeneous fluid under the assumption of the linear and time-independent temperature profile. However, for the case of transient nonlinear temperature or concentration field, the related instability has been analyzed by using the frozen-time model [13], the energy method [1,4,5,13], and the linear amplification theory [1,4,5,13]. All of these methods have a parallel history of application in the Rayleigh-Bénard convection. The first model is based on linear theory and yields the critical time as a parameter based on the quasi-static approximation. In the energy method, the generalized energy functional is derived and the parameter range at which finite disturbances will decay exponentially. The last method is an initial value model that requires the initial conditions at the time $t$=0 and the criterion to define manifest convection. Riaz et al. [3] analyzed the onset of convection in porous media under the time-dependent concentration field in self-similar coordinate. Their basic idea is quite similar to the propagation theory [14]. Recently, for the isotropic porous media, Selim and Rees [15], Hassanzadeh et al. [16] and Kim and Choi [17] reconsidered the present transient problem by employing linear stability analysis, direct numerical simulation and modified energy method, respectively. Under the linear stability theory Rapaka et al. [6] and Hidalgo et al. [7] introduced the nonmodal stability concept into the present problem. For isotropic and anisotropic media, they found the most unstable infinitesimal disturbances for a specified condition. Further historical review can be found in Nield and Bejan [18].

In the present study the onset of buoyancy-driven convection in anisotropic porous media is investigated by the relaxed energy method. Even though Ennis-King et al. [1], Hassanzadeh et al. [4], Xu et al. [5], and Hong and Kim [19] considered this problem by employing the energy method, their results are slightly different from each another. In the present work, their results are compared and extended to a larger time region. Therefore, the present work may be the complement and extension of the previous work [1,4,5,19].

THEORETICAL ANALYSIS

1. Governing Equations

The system considered here is an initially quiescent, fluid-saturated, horizontal porous layer of depth $d$, as shown in Fig. 1. Initially, the fluid layer contains no solute, i.e., $C=0$ at $t$=0. For time $t$ $\geq$ 0, the gas starts to dissolve into the fluid layer through the upper free boundary which is assumed to be maintained at uniform con-

Fig. 1. Schematic diagram of system considered here.
centration C. The lower boundary is assumed to be impermeable and at no mass flux condition. The standard governing equations for the solute-driven convective mixing consist of Darcy’s law for the fluid motion in a porous medium and the convective diffusion equation for the transport of the dissolved solute. Under the Boussinesq approximation, they can then be written as follows [1]:

\[ \nabla \cdot \mathbf{U} = 0, \]

\[ \frac{\partial \mathbf{U}}{\partial t} = -\nabla P + \rho \beta g C, \]

\[ \frac{\partial C}{\partial t} + \mathbf{U} \cdot \nabla C = \varepsilon \alpha C \nabla^2 C, \]

where U is the Darcy velocity vector, \( \mu \) is the fluid viscosity, K is the absolute permeability tensor assumed constant everywhere but not necessarily isotropic, P is the pressure, g is the gravitational acceleration, \( \varepsilon \) is the porosity C is the solute concentration, t is time, \( \beta \) is the volumetric expansion coefficient and \( \alpha \) is the effective molecular diffusivity of the solute in the aqueous phase in the porous medium. The important parameters to describe the present system are the Darcy-Rayleigh number \( Ra \) and the anisotropic ratio \( \gamma \) defined by

\[ Ra_D = \frac{g R_k C_0}{\varepsilon \alpha v}, \]

\[ \gamma = \frac{K_{\perp}}{K_{\parallel}}, \]

where \( K_{\perp} \) and \( K_{\parallel} \) denote the permeability in horizontal and vertical directions, respectively. The effect of an anisotropic permeability on the flow field was studied by Rees and Storesletten [20].

For the present transient stability analysis we define a set of nondimensionalized variables \( \tau \), \( z \), and \( C \) by using the scale of time \( d/\alpha_c \), vertical length \( d \) and concentration \( C \). Then the basic diffusion state is represented in dimensionless form by

\[ \frac{\partial C_0}{\partial \tau} = \frac{D}{\alpha_c} \frac{\partial^2 C_0}{\partial z^2}, \]

with the following initial and boundary conditions:

\[ C_0 = 0 \quad at \quad \tau = 0, \]

\[ C_0 = 1 \quad at \quad z = 0 \quad and \quad \frac{\partial C_0}{\partial z} = 0 \quad at \quad z = 1. \]

The above equations can be solved by using conventional separation of variables technique or Laplace transform method as follows:

\[ c_\circ = 1 - 2 \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \sin(\omega_n \zeta) \exp(-\omega_n^2 \tau), \]

\[ c_\circ = \sum_{n=0}^{\infty} (-1)^n \left[ \text{erfc}\left( \frac{n + 1}{2\sqrt{\tau}} \frac{z}{2}, \frac{z}{2} \right) + \text{erfc}\left( \frac{n - 1}{2\sqrt{\tau}} \frac{z}{2}, \frac{z}{2} \right) \right], \]

where \( \omega_n = (n-1/2)\pi \). Eq. (6b) converges more rapidly than Eq. (6a) for a small time region. The evolution of the basic profiles of concentration with time is described in Fig. 2. For the deep-pool region of \( \tau \leq 0.01 \), the base concentration profiles reduced:

\[ c_\circ = \text{erfc}\left( \frac{z}{2\sqrt{\tau}} \right), \]

where \( \zeta = z/\sqrt{\tau} \). The above Leveque-type solutions of Eq. (7) are in good agreement with the exact solutions of Eq. (6) in the region of \( \tau \leq 0.1 \). For \( \tau \leq 0.01 \) Eq. (6b) with \( n=0 \) yields almost the same concentration profile as Eq. (7).

**2. Stability Equation**

By perturbing Eqs. (1)-(3), we can obtain the following dimensionless equations:

\[ \nabla \cdot \mathbf{u} = 0, \]

\[ \mathbf{u} + \nabla P - Re \mathbf{c} = 0, \]

\[ \frac{\partial c}{\partial \tau} = \nabla^2 c - Rw \frac{\partial c}{\partial z} - \mathbf{u} \cdot \nabla c, \]

under the following boundary conditions:

\[ c = 0 \quad at \quad z = 0, \]

\[ c = 1 \quad at \quad z = 1. \]

where \( \nabla = \sum_{i,j,k} \frac{\partial}{\partial x_i} \mathbf{j} \mathbf{e}_j \mathbf{e}_k \) and \( \mathbf{i} \), \( \mathbf{j} \) and \( \mathbf{k} \) are the unit vectors in the Cartesian coordinate and \( R = \sqrt{Ra_p} \).

Now, multiplying Eq. (9) by \( u_j \) and Eq. (10) by \( c_i \), integrating them over the volume \( \Omega \) and then employing the divergence theorem, Eqs. (9) and (10) become the following energy identities:

\[ \frac{dE}{d\tau} = 0 = R(c_j w_j) - \frac{1}{2} \sum_{i=1}^{n} \sum_{j=0}^{\infty} \sum_{j=0}^{\infty} \text{erfc}\left( \frac{n + 1}{2\sqrt{\tau}} \frac{z}{2}, \frac{z}{2} \right) + \text{erfc}\left( \frac{n - 1}{2\sqrt{\tau}} \frac{z}{2}, \frac{z}{2} \right), \]

where \( \text{erfc}(z) = \frac{1}{2} \sum_{j=0}^{\infty} \sum_{j=0}^{\infty} \text{erfc}\left( \frac{n + 1}{2\sqrt{\tau}} \frac{z}{2}, \frac{z}{2} \right) + \text{erfc}\left( \frac{n - 1}{2\sqrt{\tau}} \frac{z}{2}, \frac{z}{2} \right), \]

where the primes have been dropped and \( \langle \rangle = \int \cdot \Omega \). In the present system the dimensionless energy functional can be defined as a linear combination of Eqs. (12) and (13) with a coupling constant \( \lambda > 0 \):

\[ E(\tau) = 0 + \frac{1}{2} \sum_{j=0}^{\infty} \sum_{j=0}^{\infty} \text{erfc}\left( \frac{n + 1}{2\sqrt{\tau}} \frac{z}{2}, \frac{z}{2} \right) + \text{erfc}\left( \frac{n - 1}{2\sqrt{\tau}} \frac{z}{2}, \frac{z}{2} \right), \]

By setting \( c_i = \mathbf{c}_i / \sqrt{\tau} \), the above energy identity can be expressed as:

\[ \frac{dE}{d\tau} = -\langle \nabla c_i \rangle \langle \mathbf{u} \rangle + R \left( w_j c_j \right) \frac{\partial c_i}{\partial z} - \sqrt{\lambda} \frac{\partial c_i}{\partial z} w_j c_i, \]

where the hats have been dropped. The above relation can be repre-