COMPLETE CLASSIFICATION OF THE POSITIVE SOLUTIONS OF $-\Delta u + u^q$

By

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Abstract. We study the equation $-\Delta u + u^q = 0$, $q > 1$, in a bounded $C^2$ domain $\Omega \subset \mathbb{R}^N$. A positive solution of the equation is moderate if it is dominated by a harmonic function and $\sigma$-moderate if it is the limit of an increasing sequence of moderate solutions. It is known that in the subcritical case, $1 < q < q_c = (N + 1)/(N - 1)$, every positive solution is $\sigma$-moderate [32]. More recently, Dynkin proved, by probabilistic methods, that this remains valid in the supercritical case for $q \leq 2$, [15]. The question remained open for $q > 2$. In this paper, we prove that for all $q \geq q_c$, every positive solution is $\sigma$-moderate. We use purely analytic techniques, which apply to the full supercritical range. The main tools come from linear and non-linear potential theory. Combined with previous results, our result establishes a one-to-one correspondence between positive solutions and their boundary traces in the sense of [36].

1 Introduction

In this paper, we study boundary value problems for the equation

$$-\Delta u + |u|^q \text{sign } u = 0, \quad q > 1$$

in a bounded $C^2$ domain $\Omega$ in $\mathbb{R}^N$. We say that $u$ is a solution of this equation if $u \in L^q_{\text{loc}}(\Omega)$ and the equation holds in the sense of distributions. Every solution of the equation is in $W^{2,\infty}_{\text{loc}}(\Omega)$. In particular, every solution is in $C^1(\Omega)$.

Let $\mathcal{M}(\partial \Omega)$ denote the space of finite Borel measures on the boundary. Set

$$\rho(x) := \text{dist}(x, \partial \Omega)$$

and denote by $L^q_{\rho}(\Omega)$ the Lebesgue space with weight $\rho$.

For $\nu \in \mathcal{M}(\partial \Omega)$, a weak solution of the boundary value problem

$$-\Delta u + |u|^q \text{sign } u = 0 \quad \text{in } \Omega, \quad u = \nu \quad \text{on } \partial \Omega$$

is a function $u \in L^1(\Omega) \cap L^q_{\rho}(\Omega)$ such that

$$-\int_{\Omega} u \Delta \phi dx + \int_{\Omega} |u|^q \text{sign } (u) \phi dx = -\int_{\partial \Omega} \partial_u \phi dv,$$
for every $\phi \in C^2_0(\bar{\Omega})$, where

\begin{equation}
C^2_0(\bar{\Omega}) := \{ \phi \in C^2(\bar{\Omega}) : \phi = 0 \text{ on } \partial \Omega \}.
\end{equation}

The boundary value problem (1.2) with data given by a finite Borel measure is well understood. It is known that if a solution exists, it is unique. Gmira and Véron [20] proved that if $1 < q < (N + 1)/(N - 1)$, the problem possesses a solution for every $\nu \in M(\partial \Omega)$; if $q \geq (N + 1)/(N - 1)$, then the problem has no solution for any measure $\nu$ concentrated at a point. The number $q_c := (N + 1)/(N - 1)$ is the critical value for (1.2). The interval $(1, (N + 1)/(N - 1))$ is the subcritical range; the interval $[(N + 1)/(N - 1), \infty)$ is the supercritical range.

In the early nineties, the boundary value problem (1.2) became of great interest due to its relation to branching processes and superdiffusions (see Dynkin [11, 12], Le Gall [24]). At first, the study of the problem concentrated on the characterization of the family of finite measures for which (1.2) possesses a solution. This question is closely related to the characterization of removable boundary sets. A compact set $K \subset \partial \Omega$ is removable if every positive solution $u$ of (1.1) which has a continuous extension to $\bar{\Omega} \setminus K$ can be extended to a function in $C(\bar{\Omega})$.

In a succession of works by Le Gall [25, 26] (for $q = 2$), Dynkin and Kuznetsov [16, 17] (for $1 < q \leq 2$), Marcus and Véron [33] (for $q \geq 2$), and [34] (providing a new proof for all $q \geq q_c$), the following results were established.

**Theorem A.** Let $K$ be a compact subset of $\partial \Omega$. Then

\begin{equation}
K \text{ is removable if and only if } C^{2/q, q'}(K) = 0.
\end{equation}

Here, $q' = q/(q - 1)$ and $C^{2/q, q'}$ denotes the $(2/q, q')$ - Bessel capacity on $\partial \Omega$.

**Theorem B.** Let $\nu \in M(\partial \Omega)$. Problem (1.2) possesses a solution if and only if $\nu \prec C^{2/q, q'}$, i.e., $\nu$ vanishes on every Borel set $E \subset \partial \Omega$ such that $C^{2/q, q'}(E) = 0$.

**Remark A.1.** For solutions in $L^q(\Omega)$, the removability criterion applies to signed solutions as well.

**Remark A.2.** For a non-negative solution $u$ of (1.1), the removability criterion can be extended to an arbitrary set $E \subset \partial \Omega$. If $u$ vanishes on $\partial \Omega \setminus \bar{E}$ (where $\bar{E}$ denotes the $C^{2/q, q'}$-fine closure of $E$), then $C^{2/q, q'}(E) = 0$ implies $u = 0$. This is a consequence of the capacitary estimates of [35]; for the definition of the $C^{2/q, q'}$ -fine topology and related concepts, see [1] and Section 4 below.

In view of the estimates of Keller [22] and Osserman [40], equation (1.1) possesses solutions which are not in $L^q(\Omega)$. In particular, for domains of class $C^2$, the