GOL’DBERG’S CONSTANTS

By

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Dedicated to the memory of A. A. Gol’dberg

Abstract. We study two extremal problems of geometric function theory introduced by A. A. Gol’dberg in 1973. For one problem we find the exact solution, and for the second one we obtain partial results. In the process, we study the lengths of hyperbolic geodesics in the twice punctured plane, prove several results about them, and make a conjecture. Gol’dberg’s problems have important applications to control theory.

1 Introduction

Gol’dberg\textsuperscript{[16]} studied a class of extremal problems for meromorphic functions. Let $F_0$ be the set of all holomorphic functions $f$ defined in the rings

$$\{z : \rho(f) < |z| < 1\},$$

omitting 0 and 1, and such that the indices of the curve $f(\{z : |z| = \sqrt[3]{\rho(f)}\})$ with respect to 0 and 1 are non-zero and distinct.

Let $U = \{z \in \mathbb{C} : |z| < 1\}$ and $F_1 \subset F_0$ be the subclass consisting of functions meromorphic in $U$. Functions in $F_1$ can be described as meromorphic functions in $U$ with the property that the numbers of preimages of 0, 1 and $\infty$, counted with multiplicities, are all finite and pairwise distinct.

Let $F_2, F_3, F_4$ be the subclasses of $F_1$ consisting of functions holomorphic in $U$, rational functions and polynomials, respectively. For $f$ in any of these classes $F_j, 1 \leq j \leq 4$, we define $\rho(f)$ as

$$\rho(f) = \sup\{|z| : f(z) \in \{0, 1, \infty\}|.$$
Gol’dberg’s constants are $A_j = \inf_{F_j} \rho(f)$, $0 \leq j \leq 4$. Gol’dberg credits the problem of minimizing $\rho(f)$ to E. A. Gorin. Gol’dberg proved that

$$0 < A_0 = A_1 = A_3 < A_2 = A_4,$$

and showed that there exist extremal functions for $A_0$ and $A_2$, but extremal functions for $A_1$, $A_3$ or $A_4$ do not exist. He also proved the estimates

$$A_0 < 0.0091 \quad \text{and} \quad 0.0000038 < A_2 < 0.0319.$$

In view of (1.1), we consider only $A_0$ and $A_2$.

The constants $A_0$ and $A_2$ are important for several reasons. They are related to the following questions.

**Problem 1.** Which triples of non-negative divisors in $U$ of finite degree are divisors of zeros, poles, and 1-points of a meromorphic function in $U$?

The constants $A_0$ and $A_2$ give the only general restrictions for this problem that are known to us.

**Problem 2.** Let $\phi_1, \phi_2, \ldots, \phi_n$ be rational functions restricted to $U$. Does there exist a meromorphic function $f$ in $U$ which avoids $\phi_1, \ldots, \phi_n$?

Avoidance means that the graphs of $f$ and $\phi_j$ are disjoint subsets of $U \times \mathbb{C}$, that is, $f(z) \neq \phi_j(z)$ for $z \in U$. If the graphs of the $\phi_j$ are pairwise disjoint, then such a function $f$ exists; this is a famous result of Slodkowski [27, Lemma 2.1]; see also [12]. If $n = 3$ and the graphs of two functions $\phi_1$ and $\phi_2$ are disjoint, then the avoidance problem is equivalent to Problem 1 for holomorphic functions [7].

The avoidance problem is important for control theory: it is equivalent to the problem of simultaneous stabilization of several single input–single output linear systems, see [7, 8, 10, 14] and references therein.

In this paper, we find the exact value of $A_0$ and some related constants which are then used in our investigation of $A_2$, on which we only have partial results.

The first explicit lower bound for $A_0$ was found by Jenkins [21] who stated his result as

$$A_0 \geq 0.00037008.$$

Blondel, Rupp and Shapiro [8] proved that $A_2 > 10^{-5}$, then Batra [5, 6] improved this to $A_2 > 0.0012$.

In Section 2, we give the precise value.