BAKER’S CONJECTURE AND EREMENKO’S CONJECTURE
FOR FUNCTIONS WITH NEGATIVE ZEROS

By
P. J. RIPPON and G. M. STALLARD

Abstract. We introduce a new technique that allows us to make progress on
two long standing conjectures in transcendental dynamics: Baker’s conjecture that
a transcendental entire function of order less than 1/2 has no unbounded Fatou
components, and Eremenko’s conjecture that all the components of the escaping
set of an entire function are unbounded. We show that both conjectures hold for
many transcendental entire functions whose zeros all lie on the negative real axis,
in particular those of order less than 1/2. Our proofs use a classical distortion
theorem based on contraction of the hyperbolic metric, together with new results
which show that the images of certain curves must wind many times round the
origin.

1 Introduction

Let \( f : \mathbb{C} \to \mathbb{C} \) be a transcendental entire function and denote by \( f^n, n = 0, 1, 2, \ldots \), the \( n \)th iterate of \( f \). The Fatou set \( F(f) \) is the set of points \( z \in \mathbb{C} \) such
that \( (f^n)_{n \in \mathbb{N}} \) forms a normal family in some neighborhood of \( z \). The complement
of \( F(f) \) is called the Julia set \( J(f) \) of \( f \). An introduction to the properties
of these sets can be found in [5].

This paper concerns two conjectures in transcendental dynamics. Eremenko’s
conjecture, arising from his paper [6] in 1989, is that the escaping set

\[
I(f) = \{ z : f^n(z) \to \infty \text{ as } n \to \infty \}
\]

has no bounded components. This conjecture has motivated much current work in
transcendental dynamics, and it has become apparent that the escaping set plays a
key role in the subject. For partial results on Eremenko’s conjecture, see [13], [20]
and [14], for example.

Baker’s conjecture, arising from his paper [2] in 1981, is that the Fatou set has
no unbounded components whenever the order of the function is less than 1/2 or

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even whenever the function has order at most 1/2 minimal type. It is known [22]
that such functions have no unbounded periodic or preperiodic Fatou components,
but it remains open as to whether such a function can have an unbounded wan-
dering domain, that is, an unbounded component $U$ of the Fatou set such that
$f^n(U) \cap f^m(U) = \emptyset$ for $n \neq m$.

Many authors have shown that Baker’s conjecture is satisfied provided some
regularity condition is imposed on the growth of the maximum modulus; but, with-
out any such condition, it is not even known whether the conjecture holds for all
functions of order zero. The strongest results in this direction are given in [10] and
in [15]. A survey of earlier work on this conjecture appears in [9].

In this paper we prove that Baker’s conjecture holds for a family of symmetric
entire functions $f$ of order less than $1/2$, without any restriction on the regularity
of the growth of $M(r)$. To do this we introduce a completely new technique based
on the winding of certain image curves. In particular, our new results cover all the
examples of functions of order less than $1/2$ that were constructed in [10] and [15]
as examples to which the techniques of those papers cannot be applied.

All existing techniques for attacking Baker’s conjecture use a “repeated stretch-
ing” technique based on the relationship between the maximum modulus and the
minimum modulus, which are defined as follows:

$$M(r) = M(r, f) = \max_{|z|=r} |f(z)|, \quad m(r) = m(r, f) = \min_{|z|=r} |f(z)|.$$

For functions of order less than 1/2, the $\cos \pi \rho$ theorem implies that $m(r)$ is rel-
tively large compared to $M(r)$ for many values of $r$. If, in addition, $M(r)$ is very
small or has a certain regularity (see [15, Theorems 4 and 5], for example), then
it can be shown that the forward images under $f$ of a long curve in an unbounded
Fatou component experience repeated radial stretching which contradicts the con-
traction property of the hyperbolic metric.

As pointed out in [15] and [17], this stretching property based on the mini-
um modulus also implies that the escaping set has a certain “spider’s web” struc-
ture, described below, which is sufficient to show that Baker’s conjecture and also
Eremenko’s conjecture hold. More precisely, this property implies that the follow-
ing subset of the escaping set has this spider’s web structure:

$$A_R(f) = \{ z : |f^n(z)| \geq M^n(R), \text{ for } n \in \mathbb{N} \},$$

where $M^n(r)$ denotes the $n$th iterate of $M$ with respect to $r$ and $R > 0$ is chosen
so that $M(r) > r$ for $r \geq R$. The set $A_R(f)$ is a subset of the fast escaping set
$A(f) = \bigcup_{n \in \mathbb{N}} f^{-n}(A_R(f))$ which has many nice properties—see [17].