Thinking wants to be Organized
Empirical Studies to the Complexity of Mathematical Thinking

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Abstract: Can the describable complexity of test problems concerning mathematical thinking and the empirical results of their dealing with be put into a relation? Can graded test problems be constructed which lead to results which can basically be predicted? Empirical studies give interesting and helpful answers which lead to didactically important consequences, just like the evaluation of the PISA results.

Key words: mathematical thinking, cognitive complexity, linguistically logical complexity, competences, standards of education.


ZDM-Classification: C30

1. The Complexity of Mathematical Thinking

Learning at school calls for the acquisition of knowledge. Knowledge should be an intelligent and applicable, not an inactive and isolated one, because knowledge is supposed to develop into ability. It is just the same when learning mathematics. It is not only about acquiring knowledge, but also and especially about developing mathematical skills and mathematical thinking. Today when school and teaching lessons are rearranged this is represented by the concept “competence”.

The concept “competence” is even a keyword. In this context competences according to Weinert (2001, p. 27f) mean cognitive abilities and skills available for individuals or learnable by them in order to solve certain problems, as well as motivational, volitional and social readiness and abilities to be able to use the problem solutions successfully and responsibly in variable situations. Competences, whose profile of demands and expectations is expressed by educational standards (Sjuts 2004a), are specifically related to the subject and the results and can, in principle, be made operational by means of problems and test scales.

When converting this into test problems some questions for competence models arise: Which models describe competences adequately and in an empirically ascertainable way? Which demand and level areas can be identified? Can test problems be undoubtedly related to certain competence levels? Is every competence level within the models characterized by cognitive processes and actions in such a way that persons of this level can cope with them, but not persons of a lower level. Developed, scientifically everywhere accepted and empirically validated competence models are obviously only recognizable in attempts so far (Helmke & Hosenfeld 2004).

The study in hand wants to look into the following two questions. First: Can the describable complexity of test problems and empirical results of their dealing with be related? Second: Can graded test problems be constructed which lead to results that can basically be predicted? The basis for the study is the complexity concept which has been developed at the Institute of Cognitive Mathematics of the University of Osnabrueck (Cohors-Fresenborg & Sjuts & Sommer 2004). This cognition theoretically orientated concept has tried to make comprehensible several characteristics that fix the degree of difficulty. On the one hand this offers the possibility to judge cognitive and metacognitive performances (Cohors-Fresenborg & Sjuts 2001), on the other hand analysing instruments which – and this is new – leads to remarkable evaluations of PISA-results and didactically significant conclusions (Cohors-Fresenborg & Sjuts & Sommer 2004).

This article picks up the characteristics “linguistically logical complexity” and “cognitive complexity”, which can be outlined as follows (Cohors-Fresenborg & Sjuts & Sommer 2004):

The characteristic “linguistically logical complexity” includes demands on the identification and understanding of a problem text (which is formed by a logical structure and linguistic interconnection), before this is transferred to mathematical description and processing.

The characteristic “cognitive complexity” includes demands on the extent, intensity and complexity of thinking processes when solving a problem – especially when the simultaneity or the interconnection of thinking steps in the solving process have to be organized in an order that has to be followed.

2. Test Problems on Mathematical Thinking: Results Part A

2.1 Description of the Test Situation

At the beginning of the school year 2004/2005 the school system in the Federal State of Lower Saxony was in a special situation. The independent “Orientierungsstufe” (the period of years 5 and 6 at school during which pupils were selected to attend different schools) had been closed. The four years at primary school were (again) followed by the structure of three types of secondary school with different demands in performance (Gymnasium – highest level, Realschule – medium level, Hauptschule – lowest level), as is the case almost everywhere in Germany. Suddenly these secondary schools had to take on not only pupils of year 7, but additionally pupils of year 5 and 6.

This was also the case at the Ubbo-Emmius-Gymnasium in Leer. Pupils for year 5 came from 30 primary schools, for year 6 and 7 from the “Orientierungstufen” that had been closed. This offered the opportunity for an extensive test. The test was to show the level of compe-
tence of the individual age-groups (and individual classes) and – at the same time – to find pupils who were worth being considered for the participation in supportive measures for mathematical competitions and in the regional co-operation for the bursary for gifted pupils.

145 pupils (of 5 classes) of year 5, 174 pupils (of 6 classes) of year 6 and 139 pupils (of 5 classes) of year 7, i.e. 458 pupils altogether took part in the test. They had to work on problems which were the same for all of them, but also on problems (approximately one third) which were different, this means three-levelled problems. The following two chapters deal with some of those problems and with the results achieved.

2.2 Analysis of the Results of the Test Problems which were the same for all Pupils

240 Euros

240 Euros are to be divided among three sons in such a way that the middle son gets 20 Euros more than his older brother and 20 Euros less than his younger brother. The eldest son gets .............., the middle son gets ............, and the youngest son gets ............

Looking at the problem superficially it is a so-called “Textaufgabe”. The correct answer is: The eldest son gets 60 Euros, the middle one 80 Euros, and the youngest one 100 Euros. 298 out of 458 pupils of all three age groups of years 5, 6 and 7 solved the problem, this corresponds to a success rate of 65.1 %.

A certain difficulty of this problem, which can also lead to mistakes in the answer, lies in the linguistically logical complexity. In this respect those cases have to be considered where test persons got the correct sum (240 Euros), but distributed it incorrectly (e.g. 20 Euros, 100 Euros, 120 Euros), or figures the sum of which was not 240 Euros. These two types of mistakes arose at rates of 30.3 % in year 5, 25.3 % in year 6 and 19.4 % in year 7.

The results can be interpreted as a lack of controlling measures on the part of the test persons. They do, however, also show that this lack of performance decreases from year to year.

Three-figure Numbers

How many three-figure numbers are there which can be formed by the three figures 1 and 2 and 3, whereby every figure may only occur exactly once.

There are ............. of such numbers.

This problem requires the ability of combinatorical thinking. 315 of the 458 test persons got the correct result (there are six of such numbers). The success rate is 68.8 %. With this problem it is not so much the linguistically logical complexity, but the cognitive one, which determines the probability of success. It is mainly about completely understanding the six combinations of the three figures. The monitoring of one’s own thinking is required.

The first statement is wrong, the other two are correct. 217 of 458 test persons marked the correct result (No) in the first statement; 209 marked Yes in the second statement; and 223 marked Yes in the third statement. 132 marked all three statements correctly. This results in a success rate of 28.8 %. A certain familiarity with the calculation of areas and circumferences is surely necessary. But apart from this the problem is also characterized by its linguistically logical complexity (every; not every, there are ..., which do not), it is therefore not surprising that the result rates to these statements hardly differ. This does also explain the total result of 28.8 %. Linguistically logical complexity is put in front of the actual dealing with the problem as an obstacle (Collors-Frenseborg & Sjuts & Sommer 2004) and thus decreases the probability of solution. When assessing the results it has, of course, to be considered that all three parts of the problem have been answered correctly.

300 Marbles

Anna has got 300 marbles. She puts half of them into a bag. Then she puts one third of the marbles in her bag aside. Martin has also got 300 marbles. He puts one third of them into a bag. He then puts one half of those marbles in his bag aside.

Mark the sentences with a cross which are correct!

- □ Anna has put more marbles aside than Martin.
- □ Martin has put more marbles aside than Anna.
- □ Anna has put as much marbles aside as Martin.

The analysis of this problem can be compared to that of the previous problem. 90 test persons marked the first possibility, 110 the second one (which are both wrong), and 219 marked the third possibility, the correct one; 33 did not make any decision at all. This is a success rate of 47.8 %. It is not so much about fractions and the use of commutativity of multiplication (one half times one third is the same as one third times one half), but about the actual thinking process of a two-step thinking and calculating process, which is expressively allowed or even suggested by mentioning a figure (300 marbles). Obviously the two stages, and thus a certain cognitive complexity, prove to be an obstacle. The calculating processes themselves (a half of 300 is 150, a third of 150 is 50 – a